

## MODELLING IN ACTION

Examining How Students Approach Modelling Real Life Situations. Three Case Studies  
(Model of the movement of an elevator)

Eugenia Marmolejo Rivas

Faculty of Science

Universidad Nacional Autónoma de México.

México

ginauco@gmail.com

### Abstract

We present two activities in which we asked students to model an elevator's movement. In the first one, students were given a model and were asked to evaluate and improve it. In the second activity, students were given neither an initial model, nor any data. The results show that when students were given an initial model, their improved model emulated the form of the initial model. When the students did not receive an initial model, they developed a variety of models which included polynomial, piecewise define function and sinusoidal functions. In this case students focused more on the movement of the elevator and on understanding its operation.

**The keywords:** Modelling, Education, Teaching, Case Study.

## 1 Introduction

How does an elevator move? To answer this question, a translation between the real context and mathematics is required, i.e, mathematical modelling is needed. We present two activities: The “Freight Elevator Activity” and the “Elevator Activity”. In the “Freight Elevator Activity” students are given a model and are asked to evaluate and improve upon it. No data are collected, therefore, in this task students conduct a part of the modelling process. In the “Elevator Activity” students do a complete modelling activity because they start with an analysis of the real life situation under study, develop a data collection strategy, and, as in the first case, they build and evaluate their models.

According to Morten Blomhøj, the general belief among teachers is that modelling activities presuppose conceptual understanding of the mathematics involved. Mathematical modelling, however, can be seen as a practice of teaching that places the relation between real life and

mathematics at the centre of teaching and learning, and this is relevant at all levels. Modelling activities may motivate learners and help them in the establishment of cognitive roots for the construction of important mathematical concepts (Blomhøj, 2004). Each activity has been designed to enrich the learning process, both support different modelling approaches and solutions, allow students to make predictions, generate explanations, validations and model comparison.

## 2 Freight Elevator Activity

### Meet Carmen

Carmen is one of the students who participated in the study. She is an 18 year old student enrolled in the Higher Level Mathematics course in the International Baccalaureate Programme; it is a two-year course. At the higher level, students carry out investigations that they analyse and discuss in the classroom. At the end of the course, they complete a final piece of work which is carried out individually, and must be completed in a maximum of five hours<sup>1</sup>. The students have access to computers with pertinent software such as Excel.

Carmen was presented with the “Freight Elevator Activity”. The key feature of this activity is that the students are given an initial model and are asked to evaluate and improve it. The activity is planned in this way to create a situation that allows the students to identify the strengths and weaknesses of the initial model, and draw conclusions about how it could be improved. The intent is for students to generate explanations and validations that ensure that the model they develop is better than the existing one. To achieve this, the students must identify the parameters that allow modification of the existing model. The instructions Carmen received were as follows (see Marmolejo, E. et al. 2012):

### Freight Elevator Activity <sup>2</sup>.

In this activity you will first explore a possible model for the motion of a freight elevator used to transport equipment and lift minerals. You will then evaluate the strengths and weakness of the model and finally create the specifications to develop your own model.

#### Analysis of the initial model

The formula  $y = 2.5t^3 - 15t^2$  represents the position  $y$  of an elevator measured in metres ( $y = 0$  represents ground level) where  $t$  represents time measured in minutes ( $t = 0$  is the starting time). We know one trip up and down, without taking into account the time to unload goods, is approximately 6 minutes and the vertical drop is approximately 100 metres.

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<sup>1</sup>It is externally moderated.

<sup>2</sup>International Baccalaureate, 2008.

Study the following displacement, velocity and acceleration graphs and use the functions to:

- a) Explain the significance of the negative, positive and zero values on the velocity graph.
- b) Explain the relationship between velocity and acceleration at those times at which the elevator is increasing velocity, decreasing velocity or is stationary.
- c) Evaluate the use and identify the problems of the initial model.

**Create your own model.**

1. Give a list of specifications to redesign the model of the elevator.
2. Create your own model. You may use one function alone or you may define a piecewise function.
3. Explain why your model satisfies the specifications of the problem and improves on the initial model.

**Apply your model.**

Explain what kind of modifications you could make to the model to be able to use it in other situations.

## **Carmen's model**

### **a) Overview**

Carmen noted that with the given model the bottom of the pit is not reached, only a depth of 80 metres is reached, and that the elevator does not stop when it returns to ground level. In order to correct this problems she considered the following family of functions:  $y = at^3 - bt^2$  and adjusted the parameters to achieve the elevator reached the bottom of the pit. She believes that the model works well the first 4.4 seconds, so she decided to take six points on this part of the graph, and reflect them with mirror line  $x = 4.4$ , obtaining eleven points. Her technique involved fitting a polynomial function using Excel.

### **b) Step-by-step analysis of Carmen's modelling approach**

Her strategy to redesign the model was as follows: by trial and error and using a graphing program<sup>3</sup>, she modified the parameters  $a$  and  $b$  of the function  $f(t) = at^3 + bt^2$  to adjust the

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<sup>3</sup>Graphing Package. Version 1.0.19. (2007). Haese and Harris Publications.

graph so as to approach the minimum of  $-100$  metres. She chose the following values for the parameters  $a$  and  $b$  :  $a = 2.3$  and  $b = -15.2$ .

She defined the following function:

$$f_1(t) = 2.3t^3 - 15.2t^2$$

(bear in mind that the initial model is  $y = 2.5t^3 - 15t^2$ ).

The minimum value of  $f_1$  is  $-98.35$  which occurs at  $4.4$  minutes. However, Carmen still faced the problem that the freight elevator does not stop when returning at ground level. To resolve this she decided to reflect the part of the graph  $f_1(t)$  between  $t = 0$  and  $t = 4.4$  using as the axis of reflection the line  $t = 4.4$ . In this way she conserved the minimum and obtained a symmetrical graph, thus ensuring the velocity at the start and end points was zero.

Her strategy was a subtle approach. She tabulated  $f_1(t)$  for the following six values of  $t$ :  $0, 1, 2, 3, 4$  y  $4.4$ , and then completed the table with symmetrical values. Her idea is recreated below:

$t$	$f_1(t)$
0	0
1	-12.9
2	-42.40
3	-74.70
4	-96
4.4	-98.35

$t$	$f_1(t)$
4.8	-96.00
5.8	-74.70
6.8	-42.40
7.8	-12.90
8.8	0.00

Her technique was then to use *Excel* to fit a polynomial to her data. The function she obtained was as follows:

$$f_2(t) = 0.0085t^6 - 0.2234t^5 + 1.9027t^4 - 4.6561t^3 - 4.218t^2 - 5.869t + 0.0191.$$

Carmen did an excellent work, her strategy is good. It is important to review the responses to avoid errors. Had she done it, she would have noticed that the elevator did not stopped at floor level, i.e. that the velocity at the beginning and end was zero; if she had plotted the function and its derivative she may have noticed this flaw. In the following figure the graphs are presented for  $f_2(t)$  and  $f_2'(t)$ , where it can be seen that the elevator only stops at the bottom, when  $t = 4.4$ , for  $f_2'(4.4) = 0$ ; but although  $f_2(0) = f_2(8.8) = 0$ , it can be seen that the derivative is not zero at these two instants. For example, if  $x = 0$ ,

$$f_2'(0) = 6(0.0085)(0)^5 - 5(0.2234)(0)^4 + 4(1.9027)(0)^3 - 3(4.6561)(0)^2 - 2(4.218)(0) - 5.869,$$

from which it follows that  $f_2'(0) = -5.869 \neq 0$ .

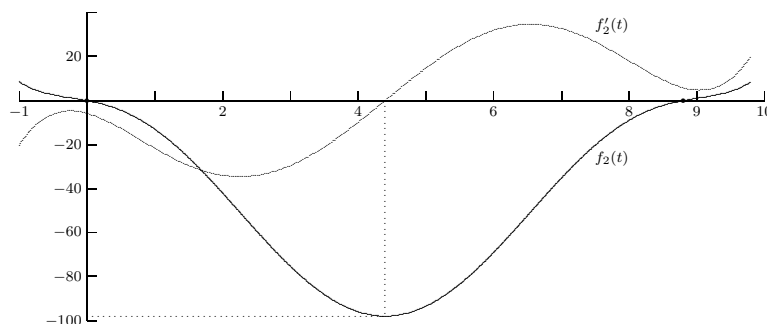


Figure 1: Graphs of position and velocity from the model proposed by Carmen.

### Remarks

Carmen did an excellent work. She chose a discrete approach to find a single function to fit her points. It is important to point out to students that when they use Excel, the software uses the specified data to fit the data, so, if we want the graph to fold, for example, in the first point of Carmen's data, if we want to have a maximum at the origin, more points to the left of the origin are needed to ensure a maximum there.

### Meet Nacho and Marc

Nacho and Marc, two of the volunteers enrolled in the first year of mathematics at a Spanish University, worked in pairs. They had access to the software package Matlab, with which they were already familiar and used to graph, derive and integrate functions (see Marmolejo, E. 2014).

### Nacho and Marc's model

#### a) Overview

From the position graph Marc and Nacho realise that the given model only reaches a depth of 80 metres and believe this was due to a built-in clearance of 20 metres. This is not a mathematical explanation, but these ideas work because there is a context. They also noted that the elevator does not stop when it returns to ground level, but continues at a velocity of 90 metres per minute (1.5 metres per second). They considered that the velocity could be modeled with a cubic function and obtained a system of four equations with three unknowns from which they obtained the parameters they needed.

#### b) Step-by-step analysis of Nacho and Marc's modelling approach

Nacho's suggestion is key for the development of the model:

Nacho: *"We should think of the points we need in order to get the most important points."*

Nacho summarized the information in a diagram in a precise and elegant way:

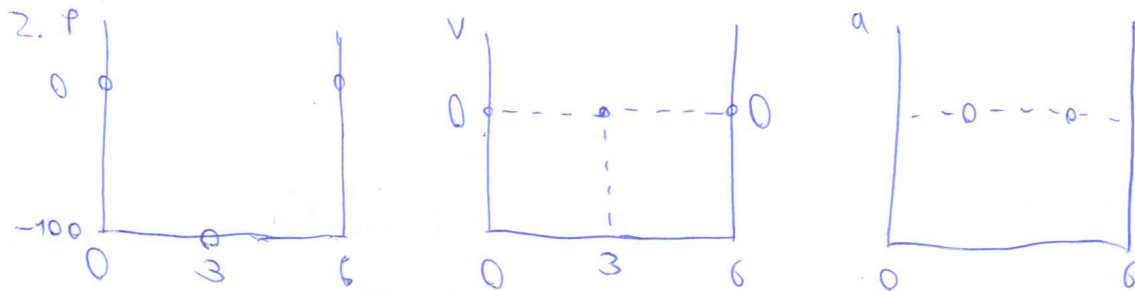


Figure 2: Important points for the position, velocity and acceleration graphs.

From the middle graph of the diagram (see Figure 2) they decided that velocity can be modeled with a cubic function, since they are thinking in terms of a polynomial which has three roots. They take into account that the independent term must be zero, since we want  $v(0) = 0$ .

Following on from the equation proposed for velocity:<sup>4</sup>  $v(t) = At^3 + Bt^2 + Ct$ , Nacho and Marc obtained, by integrating the equation for the elevator's position, and by deriving it, the equation for its acceleration.

As shown in the figure below, a system of four equations with three unknowns was obtained:

~~Vamos a mantener que el trayecto durará 6 minutos. Buscamos una función en la velocidad de la forma  $At^3 + Bt^2 + Ct = v(t)$   $y(t) = \frac{1}{4}At^4 + \frac{1}{3}Bt^3 + \frac{1}{2}Ct^2$~~

~~Queremos en  $t=3$   $At^3 + Bt^2 + Ct = 0 \rightarrow 27A + 9B + 3C = 0 \rightarrow 9A + 3B + C = 0$ .~~

~~Queremos en  $t=6$   $y(6) = 0 \rightarrow 324A + 72B + 18C = 0$~~

~~" "  $t=3$   $y(3) = -100 \rightarrow \frac{81}{4}A + 9B + \frac{9}{2}C = -100$~~

~~" "  $t=6$   $v(6) = 0 \rightarrow 216A + 36B + 6C = 0$~~

<sup>4</sup>Marc used the variable  $t$ , whereas Nacho changed it for  $x$ .

Transcript:

" We will keep the journey time at 6 minutes. We are looking for a function of velocity in  $At^3 + Bt^2 + Ct = v(t)$   $y(t) = \frac{1}{4}At^4 + \frac{1}{3}Bt^3 + \frac{1}{2}Ct^2$

We want at  $t = 3$   $At^3 + Bt^2 + Ct = 0 \rightarrow 27A + 9B + 3C = 0 \rightarrow 9A + 3B + C = 0$

We want at  $t = 6$   $y(6) = 0 \rightarrow 324A + 72B + 18C = 0$

" "  $t = 3$   $y(3) = -100 \rightarrow \frac{81}{4}A + 9B + \frac{9}{2}C = -100$

" "  $t = 6$   $v(6) = 0 \rightarrow 324A + 72B + 18C = 0$  "

What they did when Marc referred to  $t = 6$  and  $t = 3$  was to give particular values to the variable. They thus obtained a system of four equations with three unknowns.

They attempted to solve the system using Matlab but could not work out how to do it well; Fortunately they knew how to solve it by hand:

$$\begin{pmatrix} 3^3 & 3^2 & 3 & 0 \\ \frac{3^4}{4} & \frac{3^3}{3} & \frac{3^2}{2} & -100 \\ \frac{6^4}{4} & \frac{6^3}{3} & \frac{6^2}{2} & 0 \\ 6^3 & 6^2 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 27 & 9 & 3 & 0 \\ 81/4 & 9 & 9/2 & -100 \\ 324 & 72 & 18 & 0 \\ 216 & 36 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 27 & 9 & 3 & 0 \\ 216 & 36 & 6 & 0 \\ 324 & 72 & 18 & 0 \\ 81/4 & 9 & 9/2 & -100 \end{pmatrix} \sim \\
 \sim \begin{pmatrix} 27 & 9 & 3 & 0 \\ 0 & -36 & -18 & 0 \\ 0 & -36 & -18 & 0 \\ 0 & \frac{243}{4} & \frac{243}{4} & -2700 \end{pmatrix} \sim \begin{pmatrix} 27 & 9 & 3 & 0 \\ 0 & -36 & -18 & 0 \\ 0 & 0 & \frac{-48681}{16} & 164025 \end{pmatrix}$$

$$\begin{aligned}
 C &= -53'99 \\
 B &= \frac{18C}{-36} = 26'95 \\
 A &= \frac{-9B - 3C}{27} = -2'99 \\
 f &= -0'74x^4 + 8'98x^3 - 26'955x^2 \\
 &\text{Con esto solo alcanzara } h = 60.
 \end{aligned}$$

Figure 3: Nacho's solution to the system of equations

Nacho obtained the following values for the parameters:

$$C = -53.99, B = 26.95 \text{ and } A = -2.99$$

and the following function for position:

$$f(x) = -0.74x^4 + 8.98x^3 - 26.955x^2.$$

When they plot the function obtained by Nacho, they realized that this model only reached a depth of 60 metres.

Nacho: *“Look! Its less than a hundred. Why's that? Why doesn't it reach a hundred? Where did I go wrong?”*

The system is well constructed, but there was a small error in the penultimate step. The last matrix is not equivalent to the previous one.

The solutions of the system with the error corrected are:

$$C = -\frac{800}{9}, B = \frac{800}{18}, A = -\frac{400}{81}$$

From which the following functions for position, velocity and acceleration were obtained:

$$y(t) = -\frac{400}{324}t^4 + \frac{800}{54}t^3 - \frac{800}{18}t^2$$

$$v(t) = -\frac{400}{81}t^3 + \frac{800}{18}t^2 - \frac{800}{9}t$$

$$a(t) = -\frac{1200}{81}t^2 + \frac{1600}{18}t - \frac{800}{9}$$



The equations are used to graph the elevator's position (in red), velocity (in blue) and the elevator's acceleration (in green).

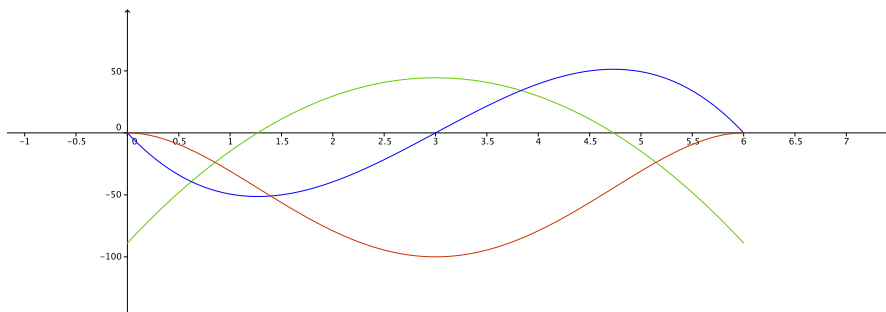


Figure 4: Graphical representation of the position (in red), velocity (in blue) and acceleration (in green) of the new model.

### Remarks

Nacho and Marc did an excellent work. They simplified the problem and were left with what really mattered: the three instants where the speed should be zero, i. e. the points at which the elevator is stationary. From there they built their model. Matlab where used to plot, derive and integrate functions, even though, and this is important, they knew how to do it by hand.

## 3 Elevator Activity

### Meet Federica

Federica is one of the students who participated. She was an 17 year old student enrolled in the Standard Level Mathematics course in the International Baccalaureate Program. At the Standard Level, students carry out investigations that they analyse and discuss in the classroom. At the end of the course, they carry out a final piece of work which they do individually, and must be completed in a maximum of five hours. The students have access to computers with pertinent software such as Geogebra.

Federica was presented with the “Elevator Activity”. The key feature of this activity is that students do a complete modelling activity because they start with an analysis of the real life situation under study, make hypothesis, develop a data collection strategy, and, as in “Freight Elevator Activity”, they build and evaluate their models.

## Elevator Activity

The objective of this activity is to model the movement of an elevator. To achieve this, you are asked to carry out the following steps:

- Make a hypothesis about how you think the graph of the position with respect to time should look for a round trip of the elevator, without taking into account the time it is stopped at the top.
- Make a hypothesis about how you think the graph of the velocity with respect to time should look for a round trip of the elevator, without taking into account the time it is stopped at the top.
- Evaluate the strengths and identify the problems of the graphs of position and velocity that you have drawn.
- Search for information on the internet. Try to answer any questions that arise.
- Make a plan to allow data collection.
- Collect data.
- Carry out analysis of the data.
- Propose a model that fits the data and specifications that are required for the elevator. You may use only one function or define a piecewise function.

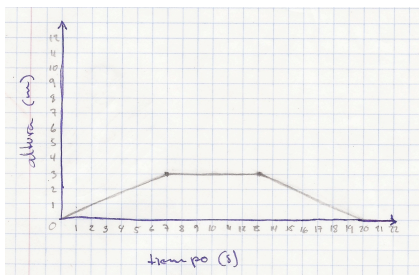
### Federicas's model

#### a) Overview

Federica made a hypothesis of the movement of the elevator. She used a graph to make a conceptual model of the position and velocity of the elevator in order to understand the physical phenomenon being studied. Then she collected the data and inputted them to the spreadsheet of GeoGebra. Federica chose a sinusoidal model as a good fit to her data. She reached this conclusion after trying and evaluating several models that she judged not to fit the data adequately. She used GeoGebra to obtain the functions she evaluated. She also used GeoGebra to adjusted the parameters to increase the accuracy of her model.

## b) Step-by-step analysis of Federica modelling approach

Federica took into consideration the time the elevation is motionless at the top. Federica's conceptual models sketching the position and velocity graphs and the comments she made about them are shown in Figures 5 and 6.

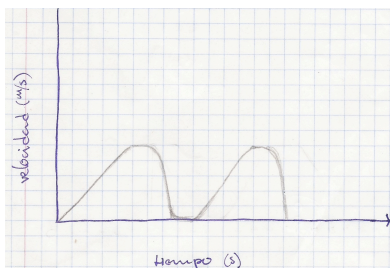


Transcript:

Federica: "It's very similar to Maria's, but mine goes up in seven seconds. Mine specifies the time. It is composed of linear graphs that showed a constant velocity and it looked like a trapezium."

Figure 5: Federica's hypothesis of position graph.

Below we show Federica's hypothesis sketch of the velocity graph:



Transcripción:

Federica: " : Its going up, then it goes like that and reaches the maximum velocity; but then it has to brake because if not it will crash into the roof. Then it arrives at zero and it doesn't matter if you get out or not, and it begins to go down."

Figure 6: Federica's graph of velocity.

Recall that the instantaneous velocity at time  $t$  is defined as the derivative of the position; note, therefore, that it may be positive or negative. The absolute value is given the name instantaneous speed. Federica's sketch clearly is one of speed (see figure 6). In order to validate the conjectures made by Federica it is necessary to study the journey of a real elevator<sup>5</sup>.

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<sup>5</sup>The data were taken in groups at the school elevator.



Figure 7: School elevator.

Federica proceeded in the following way: she marked out every 50 cm outside the school elevator with marks on adhesive tape (see figure 8).



Figure 8: Height markings.

One of her classmates went up in the elevator with a stopwatch, while she outside, advised him when the base of the elevator was passing through each mark. The data set obtained is shown in the following table:

Tieme (seg)	Height(cm)
0	0
4.4	50
7.1	100
9.9	150
13.1	200
16.3	250
19.2	298
22	250
24.8	200
27.7	150
30.7	100
33.8	50
36.9	0

Federica inputted the data to the spreadsheet page of GeoGebra and from this obtained the following graph:

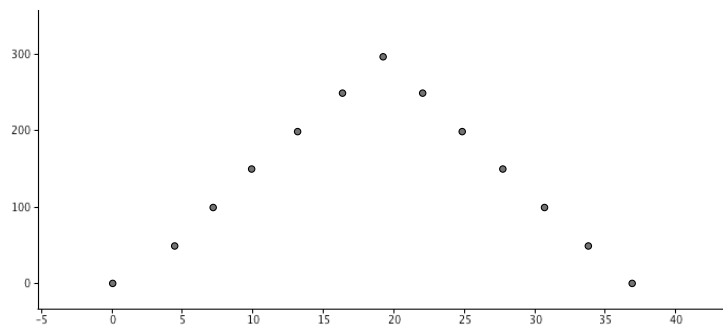


Figure 9: Graph of the data points.

GeoGebra displays the following regression model options: FitLine, FitExp, FitGrowth, FitLog, FitPoly, FitPow, FitSin. Federica tested some of them, but find that none fit good enough:

*Federica: “After plotting the points on a graph they showed an almost linear relation. However, when testing models that could fit in my graph most of the points wouldn’t touch the linear model.”*

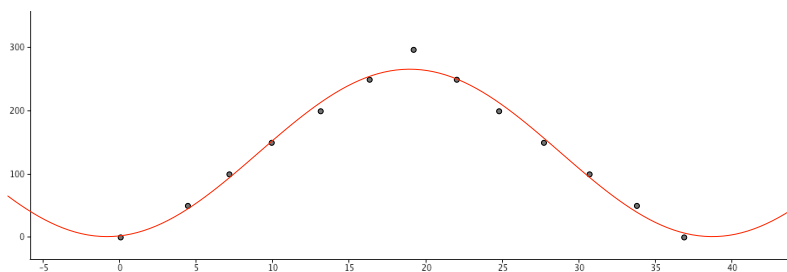


Figure 10: Sinusoidal regression model:  $y = 134.2762 + 133.1595\text{sen}(0.1586x - 1.4303)$ .

She then decided that a sinusoidal fit would be the best and considered the family of functions of the form:

$$f(x) = a \sin(bx + c) + d,$$

and determined the values of the parameters to adjust her graph, finally she proposed the following model:

$$f(x) = 149 \sin\left(\frac{1}{6}x - 1.6\right) + 149,$$

whose graph can be seen below:

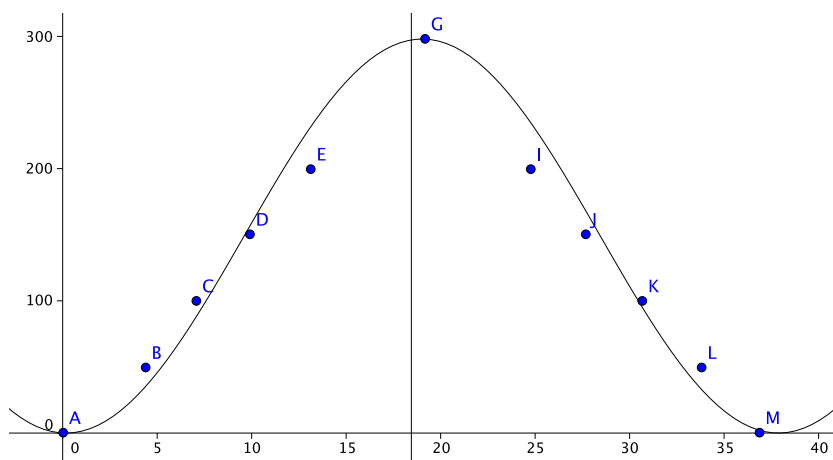


Figure 11: Sinusoidal regression model:  $f(x) = 149 \sin\left(\frac{1}{6}x - 1.6\right) + 149$ .

### Remarks

Federica did an excellent work. Using a real elevator to collect the data allowed Federica to observe the operation of the elevator. In doing so, she confirmed that the elevator travels at a constant speed most of the time, as conjectured by her. Federica was trying to find a single function to fit her points, that is way she chose a sinusoidal model.

To make the connection between the parabola and the line so that the speed change continuously, it is necessary that at the junction point of the parabola to the straight line, the tangent to the parabola has the same gradient as the straight line. In the next section we show how it could be done.

## 4 Mathematical model of the movement of an elevator

To make a fit with two functions: a parabola and a segment, for example, we need that in the connection between the parabola and the line, the speed change continuously. It is necessary that at the junction point of the parabola to the straight line, the tangent to the parabola has the same gradient as the straight line.

### How can a fit with a parabola and a line be done?

Consider the first seven data obtained in section 3:

$$(0, 0), (4.4, 50), (7.1, 100), (9.9, 150), (13.1, 200), (16.3, 250), (19.2, 298).$$

And the linear regression:  $y = 16.75x - 20.19$  obtained for the second to the sixth points (see 3). The gradient of this line is  $m = 16.75$ .

We want a parabolic fit of the form  $y = ax^2$ , passing through a point  $P = (x_0, y_0)$  on the line such that the slope of the tangent line at  $P$  is precisely  $m = 16.75$ .

That is, we need to solve the following system of three equations and three unknowns:

$$\begin{cases} y_0 = ax_0^2 \\ y_0 = 16.75x_0 - 20.19 \\ 2ax_0 = 16.75 \end{cases}$$

Solving the system we obtained:  $a = 3.47, x_0 = 2.41$  and  $y_0 = 20.15$ .

Thus,  $P = (2.41, 20.15)$ , and  $y = 3.41x^2$ , see figure 13.

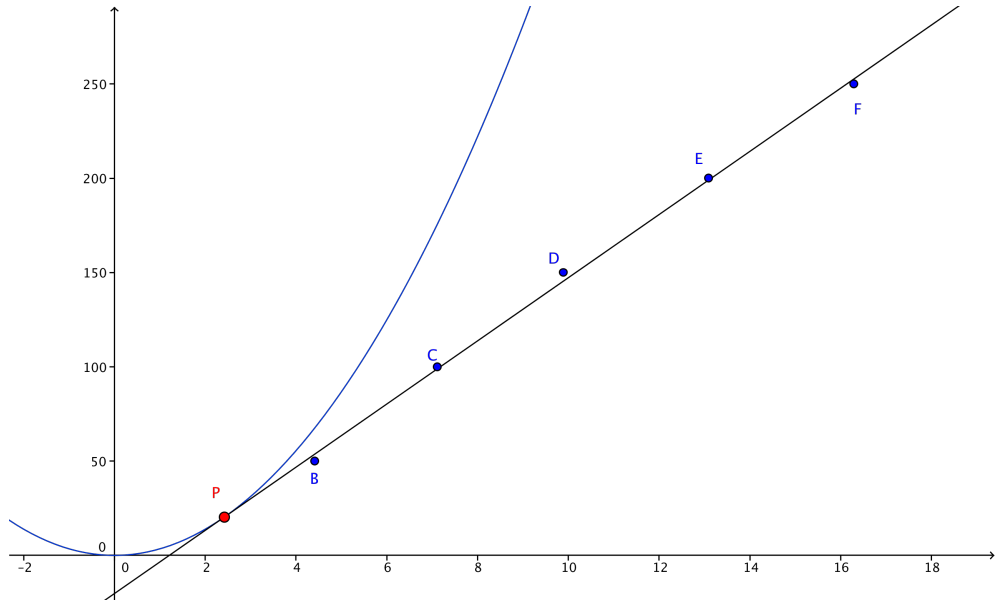


Figure 12: Piecewise regression functions.

So, the piecewise regression for this part is (see figure 13):

$$\begin{cases} y = 3.41x^2 & 0 \leq x \leq 2.41 \\ y = 16.75x - 20.19 & 2.41 \leq x \leq 16.3 \end{cases}$$

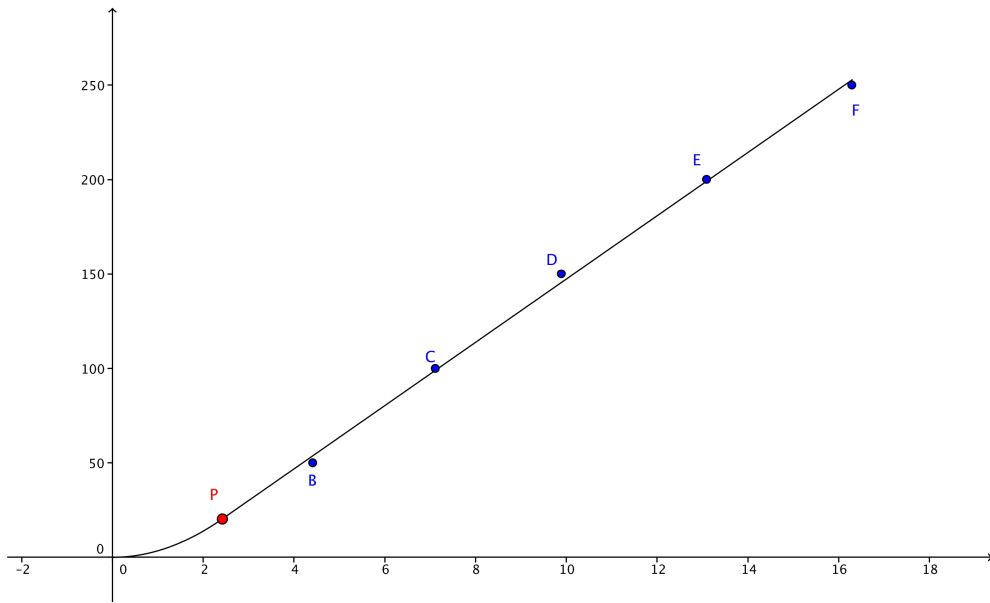


Figure 13: Piecewise regression fit.



## 5 Conclusions

This activities: the “Freight Elevator Activity” and the “Elevator Activity” places the relation between real life and mathematics at the centre of teaching and learning.

Carmen, Nacho and Marc, and Federica proposed different solutions and had different approaches for the model of the movement of the elevator. The activities allows them to make predictions, generate explanations, validations and model comparison.

Last but not least, we must emphasize that the importance of piecewise functions for modeling; and the importance of reviewing the results to avoid errors.

## 6 Thanks

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### Resources

Free software GeoGebra, available in:

<http://www.geogebra.org>