Representing Numbers As Continued Fractions
And
An N-spire .tns Document To Do Some Basic Continued Fraction Arithmetic

L Carl Leinbach
Gettysburg College
Gettysburg, Pennsylvania   17325
USA
leinbach@gettysburg.edu

Abstract: In this presentation the author presents a TI N-Spire .tns file that he created to assist teachers present some of the basic facts about continued fractions and doing arithmetic with continued fractions to students who may be studying alternative ways to present number systems, doing exact rational arithmetic, and alternatives to decimal and binary numbers. Such an approach may be a useful presentation to honors students, students who are studying alternative number systems, or as an introduction to error bounds and elementary numerical analysis.

   While the paper is a review of some of the classical works on continued factions, it does not present the material in a theorem/proof style. It motivates the results by citing examples and noting some interesting properties that may (and for completeness, need) to be followed up. Notations are given to the references where exact and complete proofs can be found. The type of investigations done in this paper are motivational and the style of presentation is designed to excite students and teachers interest in the subject.   It is the aim of the paper show that sometimes by looking at alternative representations of something that seems to be well known, such as the real numbers, can lead to exciting and useful new outcomes and applications.

    The role of the .tns document with its programs and examples is to provide a tool that remomes some of the drudgery of working with a new system and quickly produces interesting and worthwhile results. The tool is one which is popular in secondary schools and universities. It allows students quick results that they can use in their own investigations. Some of the functions have sub function calls that students can use to step through a process and check their own grasp of the algorithms necessary to do the entire calculations. Examples are given throughout the text.

The Integers

> **"**_Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk"_
> **Leopold Kronecker** (December 7, 1823 – December 29, 1891)

Given the prejudices of the times in which Kronecker lived, one can understand the sexist tone of his very famous remark. However, his point is well made. Integers are at the basis of all arithmetic and analysis. Work in this area must start with an understanding of whole numbers, or the integers as we now call them.

Actually, we humans have done a much more than respectable job with "alles andere" or "the rest".
Most of the work we have done has been using the decimal number system. It is convenient. We can generally look at two numbers in this system tell which one is larger, where they sit on the real number line and how to do arithmetic with them

$$
\begin{array}{r}
37.9 \\
+\ \ 2.5 \\
\hline
40.4
\end{array}
\qquad
\begin{array}{r}
201.4 \\
\text{x2.57} \\
\hline
14.098 \\
100.70 \\
402.8 \\
\hline
517.598
\end{array}
$$

The positioning is perfect and the operations are easily carried out with pencil and paper or within calculators.

However, there are some peculiarities that we have come to live with. For example,

$$\frac{1}{3} = 0.333333333333333\ldots\ldots\ldots$$

$$\frac{2}{3} = 0.666666666666666\ldots\ldots\ldots\ldots$$

$$1 = 1.0000000000000000\ldots\ldots\ldots\ldots$$

But, if we sum the decimal equivalents of $\frac{1}{3}$ and $\frac{2}{3}$ we get $0.9999999999999999\ldots\ldots\ldots\ldots$ We have glossed over this point by making a convention that equates the two results are.

 In order to do day to day arithmetic with the system we have built in inaccuracies caused due to rounding in order to make the system conform to the real world preference for "whole" quantities, i.e. Cheap ball point pens selling for 3 pens for 5€. The problem is that the pens cost $\frac{2}{3}$ cent more for each, if I only want one or two.

The problem is more significant than the loss of a few cents. Rounding errors can raise havoc with scientific computations. The need to round prior to beginning the computation when we first write down our approximation to irrational numbers makes error analysis a much rather complicated and difficult task. It is also hard to control the final error at the beginning of the calculation.

Example: $\frac{\pi}{e} = $?

Using the appox() operator of the TI-Nspire results in: 1.5572734979, But how much faith can be put in the result? What if our application requires more than 10 decimal places of accuracy? It seems that using the N-spire, this is the best approximation that can be made. But, we are wrong. It is possible, for example, to approximate the value of this quotient as $\frac{48982673035751402505753}{42382550732759481207436}$ which has an error of less than 10^{-39}. Furthermore, there is no other rational approximation that has a denominator of the size of this approximation or smaller that gives a better rational approximation to $\frac{\pi}{e}$. How does one determine this rational approximation and its accuracy? That is the topic of the rest of this paper.

Introduction to Continued Fractions

We start with a fairly simple rational number, say $\frac{41}{14}$ Doing integer division and rewriting this number as, we have:

$$\frac{41}{14} = 2 + \frac{13}{14} = \text{(rewriting the fractional part)}$$
$$= 2 + \frac{1}{\frac{14}{13}} = 2 + \frac{1}{1 + \frac{1}{13}}$$

This is called the regular continued fraction expansion of $\frac{41}{14}$.   Note: all of the numerators are 1. This defines our term regular.

Writing the continued fraction in this form is clumsy and requires a great deal of typographic skill as well as filling a lot of the page. There is a fortunate shorthand for regular expansions that contains all of the information that we need;

$$\frac{41}{14} = \{2;\ 1,\ 13\}$$

The first term is the floor of the original number and the rest of the terms are the floors of the reciprocals of the ensuing remainders.

The process for finding the continued fraction of any number is simply:
- At each step we have a rational number, $x > 1$
- Rewrite x as:
$$x = p + f$$
    where p is the integer part of x and f is the fractional part
- Let $x' = \frac{1}{f} > 1$ then the above equation becomes
$$x = p + \frac{1}{x'}$$
- Repeat the process for x' and continue as long as desired or until x' is an integer

Test your understanding:
   Use the above algorithm to show :

$$\frac{47}{25} = \{1;1,7,3\} \quad \frac{48}{25} = \{1;1,11,2\} \quad \frac{46}{25} = \{1;1,5,4\}$$

This process is easily translated into a TI N-spire function program:

```
Define LibPub cfrac(r,n)=
Func
:  Local i,c
:  c:={int(r)}
:  r:=r-int(r)
:  For i,1,n
:    If r>0 Then
:       r:=((1)/(r))
:       c:=augment(c,{int(r)})
:       r:=r-int(r)
:    Else
```

```
             :    Return c
             :  EndIf
             : EndFor
             : Return c
             :EndFunc
```

The reverse process, that of going from a continued fraction to a simple quotient, is even easier

```
Define LibPub cf2rat(c)=
Func
:Local i,n,r
:n:=dim(c)
:r:=c[n]
:If n>1 Then
:    For i,1,n-1
:        r:=c[n-i]+((1)/(r))
:    EndFor
:EndIf
:Return r
:EndFunc
```

Figure 1 shows the results of using the first of these two programs. The second parameter in the function call is the maximum number of places to be computed and displayed. In these examples, the process terminated before reaching this maximum.

$cfrac\left(\dfrac{46}{25},10\right)$ $\qquad\qquad$ $\{1,1,5,4\}$

$cfrac\left(\dfrac{47}{25},10\right)$ $\qquad\qquad$ $\{1,1,7,3\}$

$cfrac\left(\dfrac{48}{25},10\right)$ $\qquad\qquad$ $\{1,1,11,2\}$

$cfrac\left(\dfrac{49}{26},10\right)$ $\qquad\qquad$ $\{1,1,7,1,2\}$

Figure 1: Computing the continued fraction expansion of four rational numbers

Comparing Continued Fractions

Note the examples in Figure 1; $\frac{46}{25}$, $\frac{47}{25}$ and $\frac{48}{25}$ all differ in the third term of their continued fraction expansion. These terms are 5, 7, and 11, i.e. preserving the ordering of the original numbers. However,

$\frac{47}{25}$ and $\frac{49}{26}$ have continued fraction expansions that agree on the first three terms and differ on the fourth with values 3 and 1, respectively. Yet, $\frac{49}{26}$ is larger than $\frac{47}{25}$.

Fundamental rule for ordering numbers displayed as continued fractions:
Starting on the left, go to the first term where the expansions differ. If the position of the term is odd the sense of the inequality is preserved. If the position is even the sense of the inequality is reversed.
Continued Fractions Generate Their Own Error Estimates

Consider the rational number $\frac{638}{197} = \{3,4,5,4,2\}$. In Figure 2 we evaluate the continued fraction in stages, starting on the left. These step-by-step evaluations are called the convergents of the regular partial fraction expansion.

$cf2rat(\{3,4,5,4,2\})$	$\dfrac{638}{197}$
$cf2rat(\{3\})$	3
$cf2rat(\{3,4\})$	$\dfrac{13}{4}$
$cf2rat(\{3,4,5\})$	$\dfrac{68}{21}$
$cf2rat(\{3,4,5,4\})$	$\dfrac{285}{88}$
$3 < \dfrac{68}{21} < \dfrac{638}{197} < \dfrac{285}{88} < \dfrac{13}{4}$	true

Figure 2: Evaluating the convergents of the continued fraction for $\frac{638}{197}$.

This leads us to a very important crucial fact concerning the convergents of regular continued fraction expansions. The convergents of a continued fraction representation starting from the left satisfy the following property: (Note: The sequence is expressed as $\{a_1, a_2, a_3, a_4, \ldots, a_n\}$)

- If the subscript of the last tem of the partial sequence is odd, its rational value lies below the value of the continued fraction and above the previous odd convergent.
- If the subscript is even, its rational value lies above the value of the continued fraction and below the value of the previous even convergent.
- The odd evaluations form a monotone increasing sequence bounded above by the final value, the even ones form a monotone decreasing sequence bounded below by the final value.

THIS MEANS THAT WE HAVE A BUILT IN ERROR ANALYSIS!! The value of the continued fraction always lies between two CONSECUATIVE CONVERGENTS! Thus after determining the first two convergents we have a successively better bounds on the error of the estimate given by the value of the latest convergent. Furthermore, it has been proven (see the text by Khinchin) that each convergent is the best rational approximation that can be given having a denominator the size of the convergent or smaller.

Infinite Continued Fractions

Well, I've gotten a little ahead of myself. It is clear that the algorithm for creating a continued fraction can also be applied to irrational numbers – up to a point. Figure 3 shows a few examples.

$cfrac(e, 15)$ $\{2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1\}$

$cfrac(\pi, 12)$ $\{3,7,15,1,292,1,1,1,2,1,3,1\}$

$cfpi(87)$
$\{3,7,15,1,292,1,1,1,2,1,3,1,14,2,1,1,2,2,2,2,1,84,2,1,1,15,3,1,1,4,2,6,6,99,1,2,2,6,3,5,1,1,6\blacktriangleright$

$cfrac(\sqrt{48.4}, 20)$ $\{6.,1.,22.,3.,1.,4.,1.,1.,2.,34.,2.,1.,1.,4.,1.,7.,2.,9.,4.,1.,1.\}$

$cf2rat(\{6.,1.,22.,3.,1.,4.,1.,1.,2.,34.,2.,1.,1.,4.,1.,7.,2.,9.,4.,1.,1.\})$ 6.95701085237

$approx(\sqrt{48.4})$ 6.95701085237

$convergents(\{6.,1.,22.,3.,1.,4.,1.,1.,2.,34.,2.,1.,1.,4.,1.,7.,2.,9.,4.,1.,1.\}, 21)[1 \quad 21]$
                                                                  6.95701085237

$convergents(\{6.,1.,22.,3.,1.,4.,1.,1.,2.,34.,2.,1.,1.,4.,1.,7.,2.,9.,4.,1.,1.\}, 21)[2 \quad 21]$
                                                                  1.45656577803E-19

Figure 3: Some examples of partial regular expansions of continued fractions of some irrational numbers

The first example is the first 15 terms of the regular expansion for **e**, the base for the natural logarithms. The second is the first 12 terms of the regular expansion for **Pi**. In both cases, the length of the expansion is confined to the limit that the N-spire will allow the author's program **cfrac( )** to go without generating an error. However, note the pattern in the regular expansion for **e**. This pattern is exploited in the author's program **cflnbase(n)**, which will allow one to compute as many places of the continued fraction for **e** as the user desires. There are several other irrational numbers that have nice expansions.

The regular continued fraction expansion of **Pi** is a different matter. There is no known apparent pattern in the regular expansion of **Pi**. The author's program, **cfpi(n)** uses a partial regular expansion that is 87 places long and was found at the web site http://www.math.uiuc.edu/~hildebr/453.spring11/pi-cf.pdf .
Longer expansions do exist, but unfortunately progress is only made on this front as more and more accurate decimal expansions are found. Thus, if you are using the author's .tns document, the result is only valid for n ≤ 87. It should be noted that if one is not confined to regular expansions, there are non-

regular (numerators other than one) continued fraction expansions for **Pi** that exhibit patterns. Unfortunately, these expansions do not conform to the arithmetic algorithms given later in this paper.

The Regular Continued Fraction for the Square Root of a Natural Number

Figure 4 illustrates an important fact concerning the regular continued fraction expansion of square roots of a natural number.

$cfrac\left(\sqrt{1000},20\right)$ $\{31,1,1,1,1,1,6,2,2,15,2,2,6,1,1,1,1,1,62,1,1\}$

$cfrac\left(\sqrt{48},10\right)$ $\{6,1,12,1,12,1,12,1,12,1,12\}$

$cfrac\left(\sqrt{7},10\right)$ $\{2,1,1,1,4,1,1,1,4,1,1\}$

$cfrac\left(\sqrt{7},20\right)$ $\{2,1,1,1,4,1,1,1,4,1,1,1,4,1,1,1,4,1,1,1,4\}$

$cfsqrt\left(\sqrt{76},40\right)$
$\{8,1,2,1,1,5,4,5,1,1,2,1,16,1,2,1,1,5,4,5,1,1,2,1,16,1,2,1,1,5,4,5,1,1,2,1,16,1,2,1,1\}$

$cfsqrt\left(\sqrt{1047},40\right)$
$\{32,2,1,3,1,20,1,3,1,2,64,2,1,3,1,20,1,3,1,2,64,2,1,3,1,20,1,3,1,2,64,2,1,3,1,20,1,3,1,2,64\}$

Figure 4: Regular Continued Fraction Expansions of Some Square Roots

A close examination of these results will show that after the first integer, all of these expansions seem to be periodic. This result can be easily proven, but more instructive and easy to understand is the idea of the proof as shown in course notes from Surrey Univerity, "Introduction to Continued Fractions", by R. Knott. We will illustrate with the $\sqrt{8} = \{2, 1, 4, 1, 4, 1, 4, \ldots\}$.

> *We start our continued fraction by choosing n such that $n^2 < 8$*
> *So we have:*
> $$\sqrt{8} = 2 + \frac{1}{x1} \text{ (subtract 2 from both sides and invert)}$$
> $$x_1 = \frac{\sqrt{8}+2}{4} \text{ (obtained by rationalizing the denominator)} = 1 + \frac{1}{x2}$$
> $$x_2 = \frac{4}{\sqrt{8}-2} = \frac{4(\sqrt{8}+2)}{4} = \sqrt{8}+2 = 4 + \frac{1}{x_1}$$
> *Thus, we are back where we started and the process repeats.*

Knott did the same type of calculation using $\sqrt{5}$ as the example. This takes a few more steps to reach the same point where the pattern repeats.

The author's N-spire program **cfsqrt(*r*,*n*)** will return the first *n* terms of the infinite continued fraction for *r* where *r* is the square root of a natural number. It calls a program, **oneperiod(*r*,*m*)**,

that finds the periodic part of the expansion. The parameter m sets a bound on the length of the period.

Error Analysis and Some Other Facts

We have seen that continued fractions have a built in upperbound on the error of summing their partial values. It is the magnitude of the difference of the last two convergents, i.e.

if $\alpha = \{a_0, a_1, a_2, \ldots\}$

and p_k/q_k is the kth convergent

then

$$| \alpha - p_k/q_k| \leq | p_{k-1}/q_{k-1} - p_k/q_k |$$

But there is also a lower bound on the error

$$| \alpha - p_k/q_k| \geq 1/ q_k(q_{k+1} + q_k)$$

These results can be found in the first chapter of the text by A. Ya. Khinchin.

There is an easy rule for finding the inverse of a number expressed as a regular expansion of a continued fraction, $\{a_0, a_1, a_2, \ldots\}$. If $a_0 \geq 1$, simply move all terms one place to the right and put 0 in the initial position , i. e. If $\alpha = \{a_0, a_1, a_2, \ldots\}$ then

$$\frac{1}{\alpha} = \{0, a_0, a_1, a_2, \ldots\}$$

Conversely, if $a_0 = 0$, then move all terms one space to the left. The proofs of these results are trivial if you look at the basic definition of the regular expansion.

If we consider $0 = \frac{1}{\infty}$, then the regular expansion for 0 is $\{\infty, \ldots\}$. This representation will play a role in what follows

Rational numbers all have a finite regular expansion. Irrational numbers all have infinite regular expansions.

There is a one-to-one correspondence between the real numbers and their regular expansion. This is not quite true. There is a slight problem with the rational numbers. For example:

$$\frac{22}{9} = \{2, 2, 4\} = \{2, 2, 3, 1\}$$

 This anomaly can be easily fixed if we impose a strict stopping rule that the expansion stops when the remainder, after taking the floor in the function **cfrac()**, is of the form $\frac{1}{k}$, where k is a positive integer.

Doing Arithmetic with Regular Continued Fraction Expansions

For a long time, it was thought that while continued fraction expansions had nice properties, they had the deficiency that doing arithmetic with them was quite impossible. Of course, it was always possible to convert the continued fractions to rational expressions, do the arithmetic, and then

convert back. But, this cuts the legs right out from under the regular expansion. As will be seen, the problem was our right handed bias for doing arithmetic in the decimal system.

In 1973 William Gosper began an unpublished, but widely distributed paper, "Contrary to everybody, this self-contained paper will show that continued fractions are not only amenable to arithmetic, they are amenable to perfect arithmetic." The backbone of Gosper's paper is the homographic transformation, $\frac{a+bx}{c+dx}$, and its two variable analog, $\frac{axy+bx+cy+d}{exy+fx+gy+h}$, the bi-homographic transformation. The presentation here is based on the reference by M. J. Dominus.

We first consider the equation for the homographic transformation:
$$z = \frac{a+bx}{c+dx} \qquad a, b, c, \text{ and } d \text{ integers}$$
This is commonly abbreviated as the matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. For $x \geq 0$, this function takes on values lying between $\frac{a}{c}$ and $\frac{b}{d}$. This can be seen in Figure 5. Of course, the concavity and position of the graph depencd on $a, b, c,$ and d. In any case, the coefficients of the matrix give us our first bounds on the value of z.
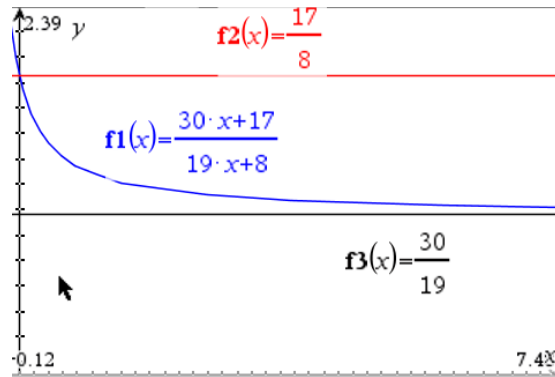
Figure 5: An illustration that the graph of a Homographic
Function lies between the values $\frac{a}{c}$ and $\frac{c}{d}$

The lesson learned from the example in Figure 5 is that if $\frac{a}{c}$ and $\frac{b}{d}$ have the same integer parts, then we know the first digit of the continued fraction for z. Consider the input from the continued fraction represented by x. Suppose that its value is p. Then
$$x = p + 1/x'$$
$$\therefore \; z = \frac{b + (a + bp)x\prime}{d + (c + dp)x'}$$
Thus, after an input of p from the continued fraction for x, the matrix corresponding to z is:
$$\begin{bmatrix} b & a + bp \\ d & c + dp \end{bmatrix}$$
Suppose that the integer parts of the ratios $\frac{a}{c}$ and $\frac{b}{d}$ both equal q. Then we know that
$$z = q + \frac{1}{z\prime} \quad \text{or} \quad z' = \frac{1}{(z-q)}$$

Thus,

$$z' = (\frac{a+bx}{c+dx} - q)^{-1} = \frac{c+dx}{a-cq+(d-bq)x}$$

and the resulting matrix is:

$$\begin{bmatrix} c & d \\ a-cq & d-bq \end{bmatrix}$$

At this point, we can write a computer program that inputs values from x until the ratios $\frac{a}{c}$ and $\frac{b}{d}$ are both equal and then outputs values for the continued fraction representing z.

There is one slight complication to the process: the matrix for z may be capable of more output after the input from x is exhausted. In that case we input ∞ (remember the continued fraction for 0) until the denominator of the matrix becomes [0,0]. This is done in the author's program, **hgraphic(*cf,m*)** that has parameters of a continued fraction, *cf,* and the transformation matrix, *m.*

We will evaluate

$$z = \frac{17+30x}{8+19x} \quad \text{for} \quad x = \frac{13}{11} = \{1,5,2\}$$

In Figure 6, we show the result of the computation.

$$\frac{30 \cdot \frac{13}{11} + 17}{19 \cdot \frac{13}{11} + 8} \qquad\qquad\qquad\qquad \frac{577}{335}$$

$$hgraphic\left(\{1,5,2\}, \begin{bmatrix} 17 & 30 \\ 8 & 19 \end{bmatrix}\right) \qquad\qquad\qquad \{1,1,2,1,1,1,1,18\}$$

$$cf2rat(\{1,1,2,1,1,1,1,18\}) \qquad\qquad\qquad \frac{577}{335}$$

Figure 6: The result of the program *hgraphic* ($\{1,5,2\}, \begin{bmatrix} 17 & 30 \\ 8 & 19 \end{bmatrix}$) and a check of the accuracy

At this point we have solved one problem in arithmetic involving continued fractions. We can add a rational number to a regular continued fraction expansion. For example, suppose that we wish to add

$$\frac{3}{4} + \{1,3,9,2\} = \frac{3}{4} + x$$

Putting everything over a common denominator of 4, we have the homographic expression

$$\frac{3+4x}{4+0x}.$$

However, the major contribution of Gosper's 1972 paper is the introduction of the bihomographic function and its corresponding matrix

$$z = \frac{axy+bx+cy+d}{exy+fx+gy+h} \qquad M = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \qquad x,y \text{ are regular continued fractions.}$$

The process for finding z, given continued fractions *x* and *y* is remarkably similar to evaluating the homographic function in one varialble.

As one would expect, the major decisions to be made are: when do we input from *x*; when do we input from *y*; and when and what do we output for *z* ? If we proceed as we did in the case of a Homographic Fuction, our basic input operations become:

inputting a *p* from *x* results in the new matrix $\begin{bmatrix} b & a+bp & d & c+dp \\ f & e+fp & h & g+hp \end{bmatrix}$

inputting a *q* from *y* results in the new matrix $\begin{bmatrix} c & d & a+cq & b+dq \\ g & h & e+gq & f+hq \end{bmatrix}$

outputting an *r* from *z* yields the new matrix $\begin{bmatrix} e & f & g & h \\ a-er & b-fr & c-gr & d-hr \end{bmatrix}$

We stop the process when after output the bottom row is [0, 0, 0, 0]. If this condition is not achieved and the input from *x* or *y* or both is exhausted and an input is called for we input ∞ (recall the regular continued fraction representation of 0). But what about our three questions of: input from *x*, input from *y*, and output to *z*?

The question of when to output is the easier of the two to answer. We look at the integer values (floor) of the ratios: $\frac{a}{e}, \frac{b}{f}, \frac{c}{g}$ and $\frac{d}{h}$. If they are all equal, i.e. they agree on the integer value to be output – their common value, say $\left\lfloor \frac{a}{e} \right\rfloor$.

Now, we come to the question of input. Once again, we rely on the ratios: $\frac{a}{e}, \frac{b}{f}, \frac{c}{g}$ and $\frac{d}{h}$. This time we look at the magnitude of their differences $d_1 = \mathrm{Max}(|\frac{a}{e} - \frac{c}{g}|, |\frac{b}{f} - \frac{d}{h}|)$ and $d_2 = \mathrm{Max}(|\frac{a}{e} - \frac{b}{f}|, |\frac{c}{g} - \frac{d}{h}|)$. If $d_1 > d_2$ and *x* still has non zero input to use, input from x. Otherwise, if *y* still has non zero input to use, input from y. If both have used their non zero values, input ∞ for x or y according to the truth of the inequality $d_1 > d_2$.

In conclusion, we do a multiplication example of the program **cf_arith(*cf1, cf2, m*)**. This is shown in Figure 7. The parameters are the two continued fractions to be multiplied and the bi-homographic matrix for multiplication.

Figure 7: An example of the cf_arith function and a check of the result

REFERENCES

1. Dominus, M.J., <u>Arithmetic with Continued Fractions,</u> unpublished lecture slides available at
   http://www.plover.com/~mjd/cftalk

1a. Dominus, M.J., Companion program to [1] at
   https://github.co/mjdominus/cf/blob/master/cf_arith.c

2. Gosper, W., <u>Continued Fraction Arithmetic</u>, unpublished (but widely circulated) paper.
   available at http://home.strw.leidenuniv.nl/~gurkan/gosper.pdf

3. Khinchin, A. Ya., <u>Continued Fractions</u>, Dover Publications, INC., Mineola, N. Y.

4. Knott, R., <u>An Introduction to Continued Fractions</u>, unpublished
   available at http://www.maths.surrey.ac.uk/hosted-sites / R.Knott/Fibonacci/cfINTRO.htm

The TI N-spire document containing the programs mentioned in this article are available by contacting the author at the e-mail address given in the heading to the article.