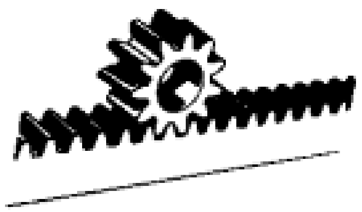
TIME 2014

1. 7. – 5. 7. Krems a. d. Donau

TECHNICAL PROBLEMS

SOLVED BY SEC2 MATHEMATICS



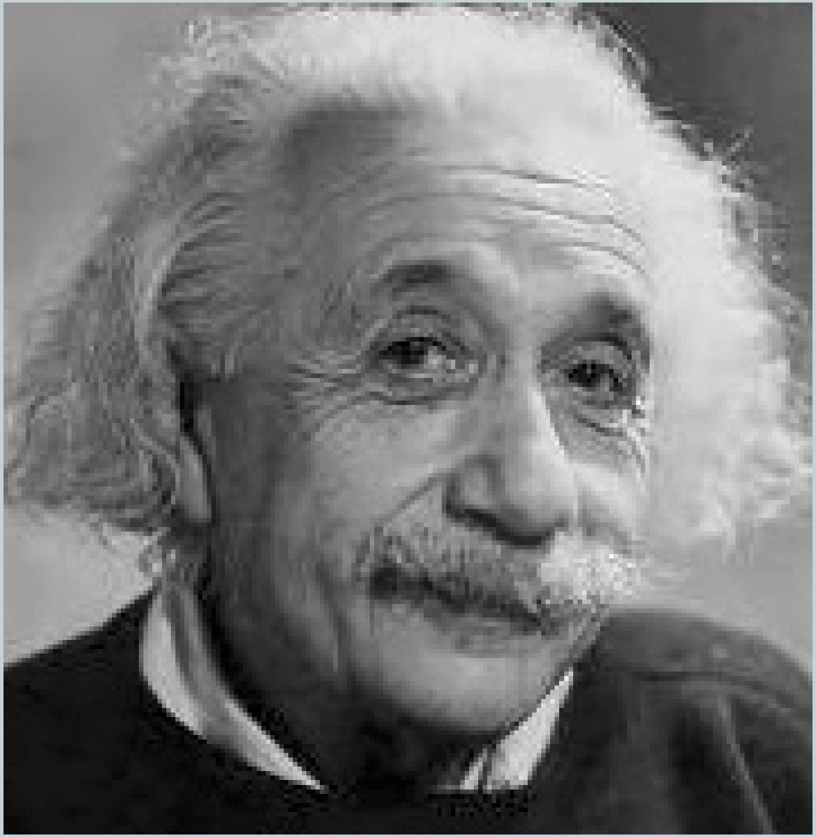
Spur gear /gear rack

$$\vec{f}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} r \cdot t - r \cdot \sin(t) \\ r - r \cdot \cos(t) \end{pmatrix}$$

Wolfgang Alvermann

BBS II Emden

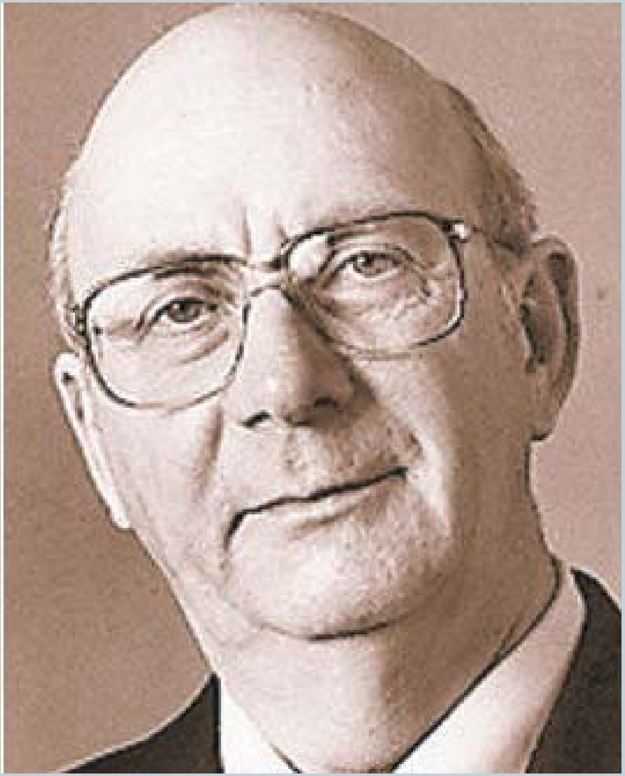
„How can it be, that mathematics, being after all a subject of human thought, independent of experiences, is so admirably adapted to the objects of reality?“



Technical problems– solved by Sec2 mathematics

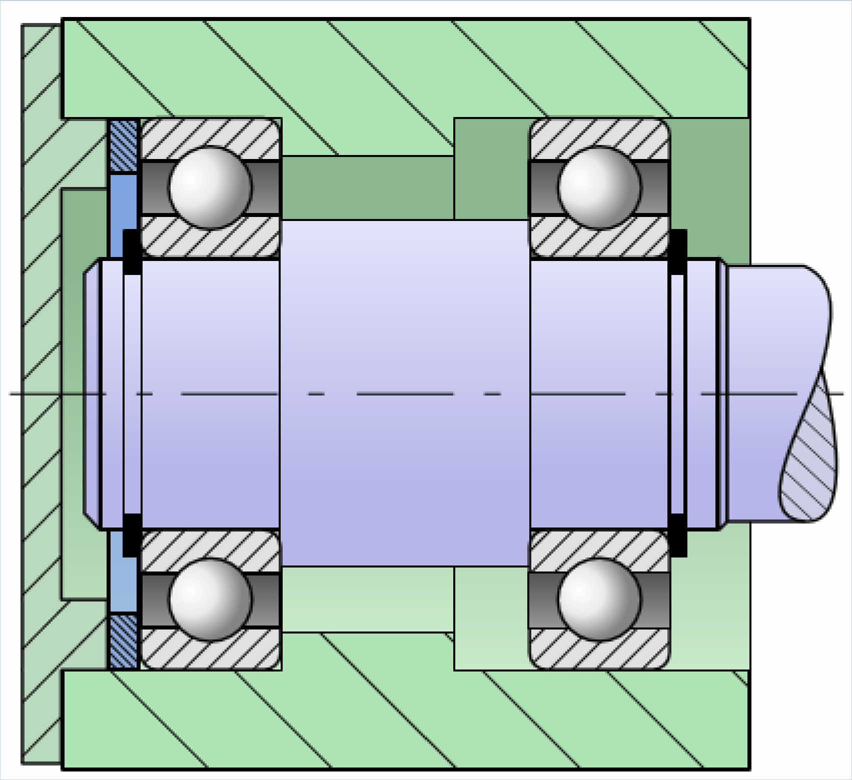
I tread the most beautiful and most difficult problems during my teaching, the less difficult I pose in tests and the easiest problems are appropiate for the end of examinations.

Prof. G. Steinberg



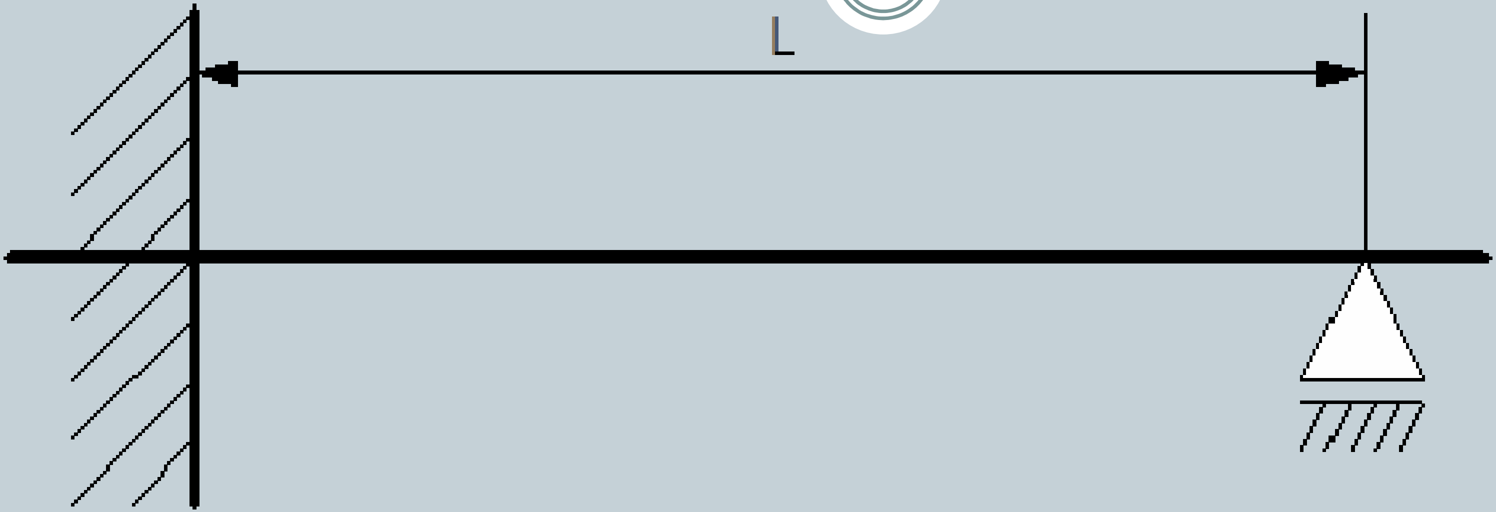
Deflection curves - examples

Bearing of a shaft



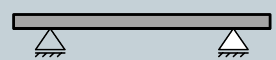
Trolley

Deflection curves

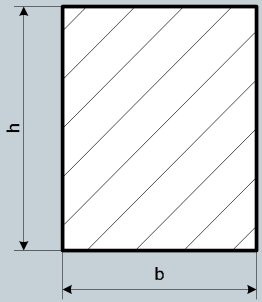
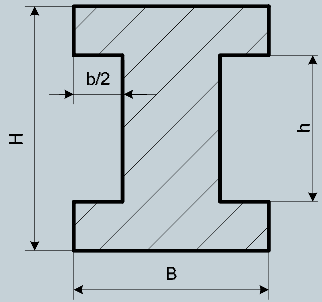
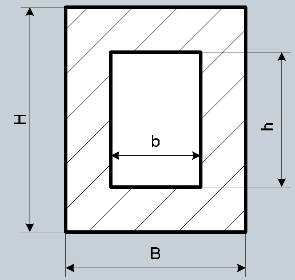
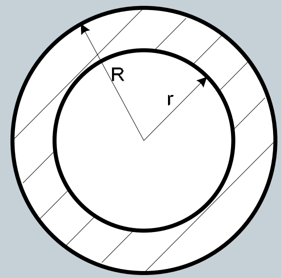


- Outline the approximate course of the deflection curve in the given drawing!
- Find out the functional equation of the deflection curve!
- Calculate the coordinate x_2 where the maximum sag appears!
- Find the equation of the tangent line in the floating bearing!
- Determine the point of inflection of the deflection curve within the internal range of [0;L]!

Deflection curves - conditions

Sag by forces perpendicular to the beam	**Sag (deflection curve)**	$w(x)=\int w'(x)dx$	w in [m]
	Gradient	$w'(x)=\frac{1}{E\cdot I}\cdot\int M(x)dx$	w'(x) without a unit
	Static moment	$w''(x)=\frac{1}{E\cdot I}\cdot(-M(x))$	M in [Nm]
	Shearing force	$w'''(x)=\frac{1}{E\cdot I}\cdot(-Q(x))$	Q in [N]
$E_{steel}=210\cdot10^9\,\frac{N}{m^2}$	**Line load**	$w''''(x)=\frac{1}{E\cdot I}\cdot q(x)$	q in $\left[\frac{N}{m}\right]$
$E_{wood}=9\cdot10^8\,\frac{N}{m^2}$		**Area moment of inertia**	I in [m^4]
		Modulus of elasticity	E in $\left[\frac{N}{m^2}\right]$
Conditions for the bearing	**Clamp**	$w(x)=0$ $w'(x)=0$	
	Bearing	$w(x_{bearing})=0$ $w''(x_{bearing})=0$	

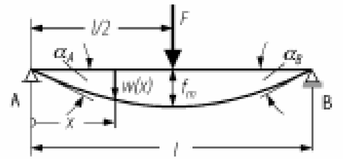
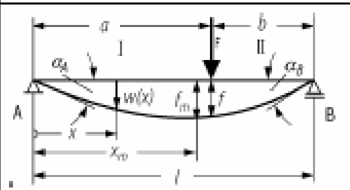
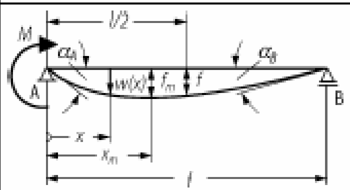
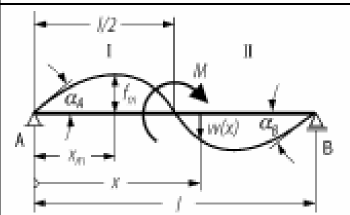
Deflection curves - beams

Rectangularly beam	Double T□girder	□ollow section	Annulus section
			
$I = \frac{b \cdot h^3}{12}$	$I = \frac{B \cdot H^3 - b \cdot h^3}{12}$	$I = \frac{B \cdot H^3 - b \cdot h^3}{12}$	$I = \frac{\pi}{4} \cdot \left(R^4 - r^4\right)$

The area moment of inertia I is a characteristic variable; it determined the retarding force against the sag.
The retarding force depends on the form of the beam.
$E \cdot I$ is named bending stiffness or flexural rigidity

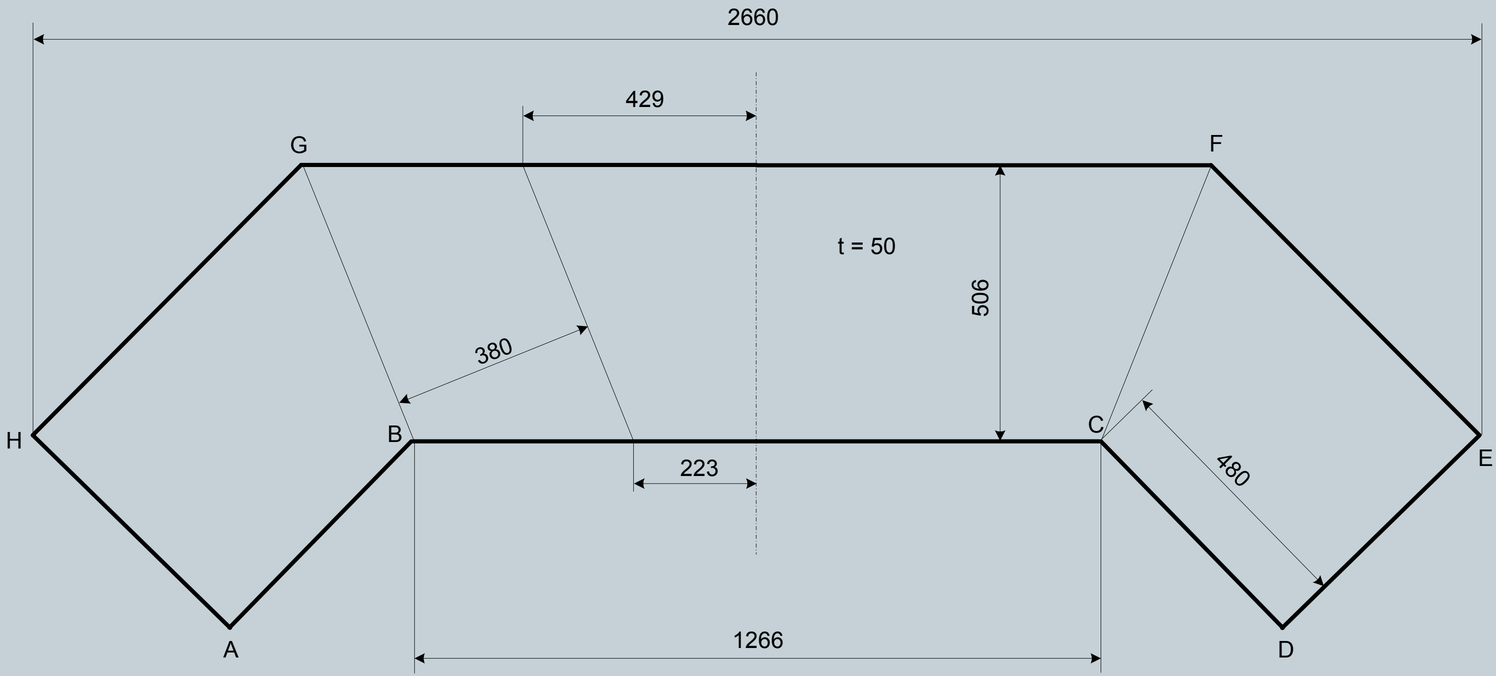
Deflection curves - worksheet

Tabelle 5a. Biegelinien von statisch bestimmten Trägern mit konstantem Querschnitt

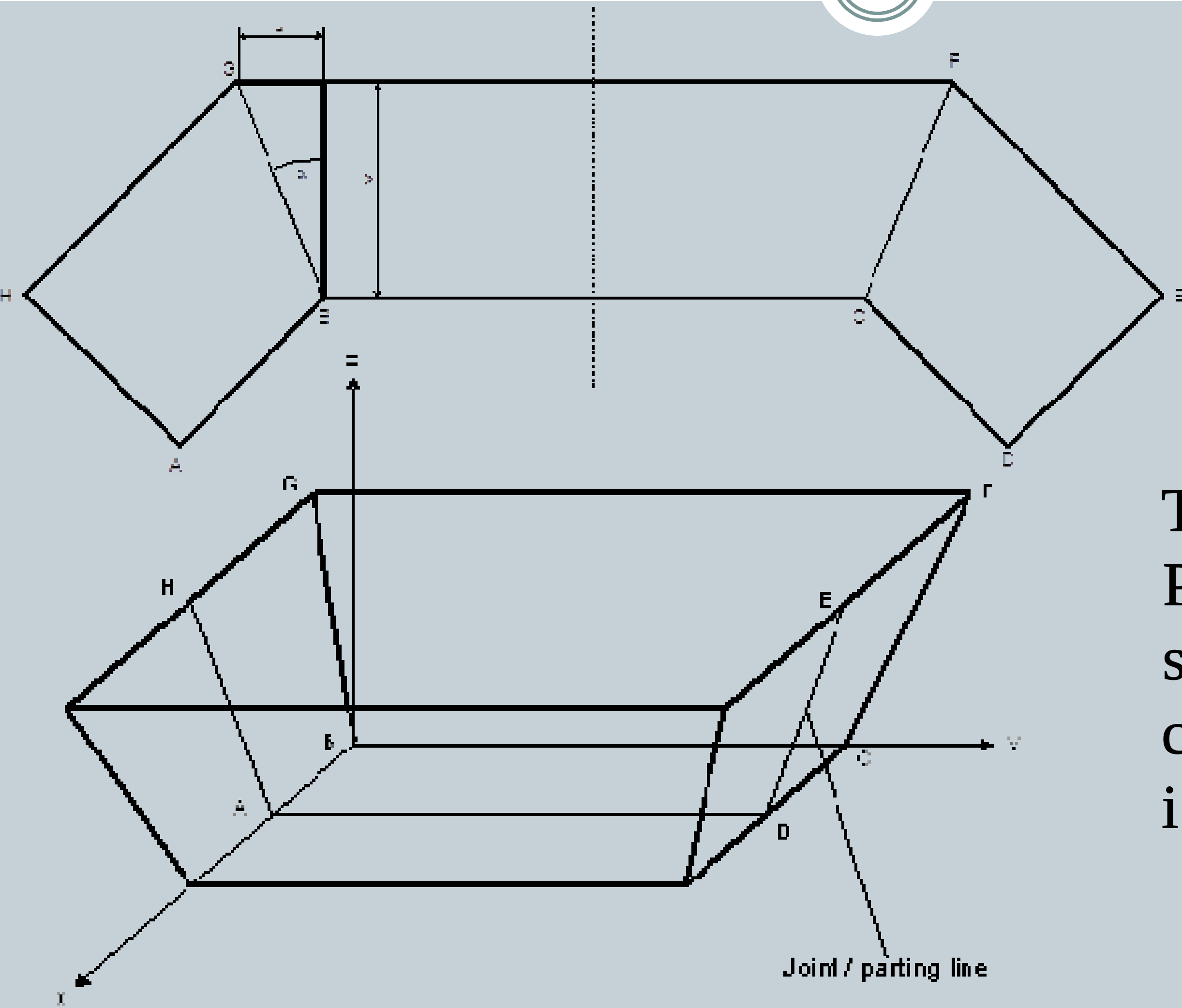
	Belastungsfall	Gleichung der Biegelinie	Durchbiegung	Neigungswinkel
1		$0 \le x \le l/2$: $w(x) = \frac{Fl^3}{48EI_y}\left[3\frac{x}{l} - 4\left(\frac{x}{l}\right)^3\right]$	$f_m = \frac{Fl^3}{48EI_y}$	$\alpha_A = \alpha_B = \frac{Fl^2}{16EI_y}$
2		$0 \le x \le a$: $w_I(x) = \frac{Fab^2}{6EI_y}\left[\left(1+\frac{l}{b}\right)\frac{x}{l} - \frac{x^3}{abl}\right]$ $a \le x \le l$: $w_{II}(x) = \frac{Fa^2b}{6EI_y}\left[\left(1+\frac{l}{a}\right)\frac{l-x}{l} - \frac{(l-x)^3}{abl}\right]$	$f = \frac{Fa^2b^2}{3EI_y l}$ $a > b: f_m = \frac{Fb\sqrt{(l^2-b^2)^3}}{9\sqrt{3}EI_y l}$ in $x_m = \sqrt{(l^2-b^2)/3}$ $a < b: f_m = \frac{Fa\sqrt{(l^2-a^2)^3}}{9\sqrt{3}EI_y l}$ in $x_m = l - \sqrt{(l^2-a^2)/3}$	$\alpha_A = \frac{Fab(l+b)}{6EI_y l}$ $\alpha_B = \frac{Fab(l+a)}{6EI_y l}$
3a		$w(x) = \frac{Ml^2}{6EI_y}\left[2\frac{x}{l} - 3\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3\right]$	$f = \frac{Ml^2}{16EI_y}$ in $x = \frac{l}{2}$ $f_m = \frac{Ml^2}{9\sqrt{3}EI_y}$ in $x_m = l - \frac{l}{\sqrt{3}}$	$\alpha_A = \frac{Ml}{3EI_y}$ $\alpha_B = \frac{Ml}{6EI_y}$
3b		$0 \le x \le l/2$: $w_I = \frac{Ml^2}{24EI_y}\left[-\frac{x}{l} + 4\left(\frac{x}{l}\right)^3\right]$ $l/2 \le x \le l$: $w_{II} = \frac{Ml^2}{24EI_y}\left[-3 + 11\frac{x}{l} - 12\left(\frac{x}{l}\right)^2 + 4\left(\frac{x}{l}\right)^3\right]$	$f_{mI} = f_{mII} = \frac{Ml^2}{72\sqrt{3}EI_y}$ in $x_{mI} = \frac{l}{2\sqrt{3}}$ bzw. in $x_{mII} = l\left(1 - \frac{1}{2\sqrt{3}}\right)$	$\alpha_A = \alpha_B = \frac{Ml}{24EI_y}$

A bending Problem

- The material has to be bent along the edges BG or CF in such a way that the edges from AB and BC and BC and CD stick together vertically.



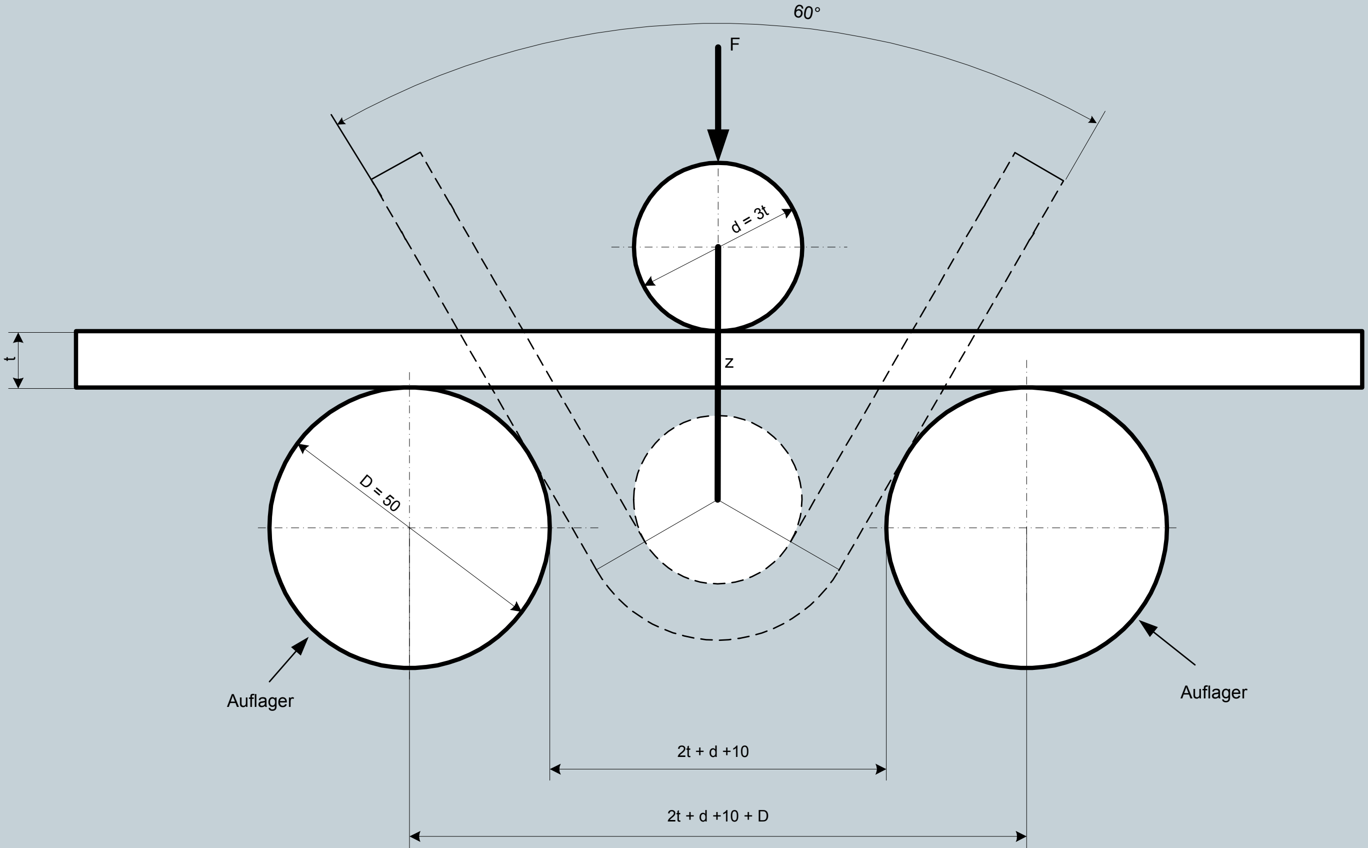
A bending Problem



The coordinates of Point G in a three-d system of coordinates are important.

A bending test

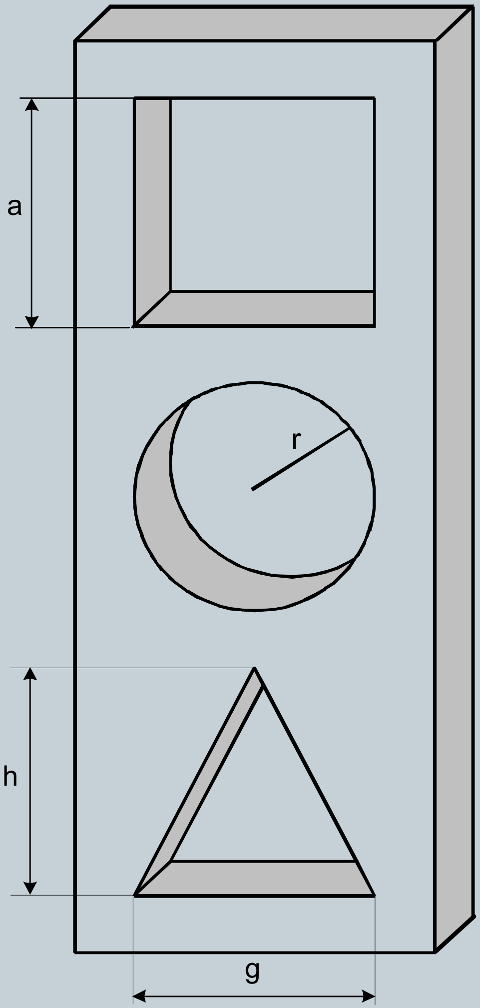
Which distance z (in mm) must the bending ram be moved to bend the material in an angle of 60 ° ?



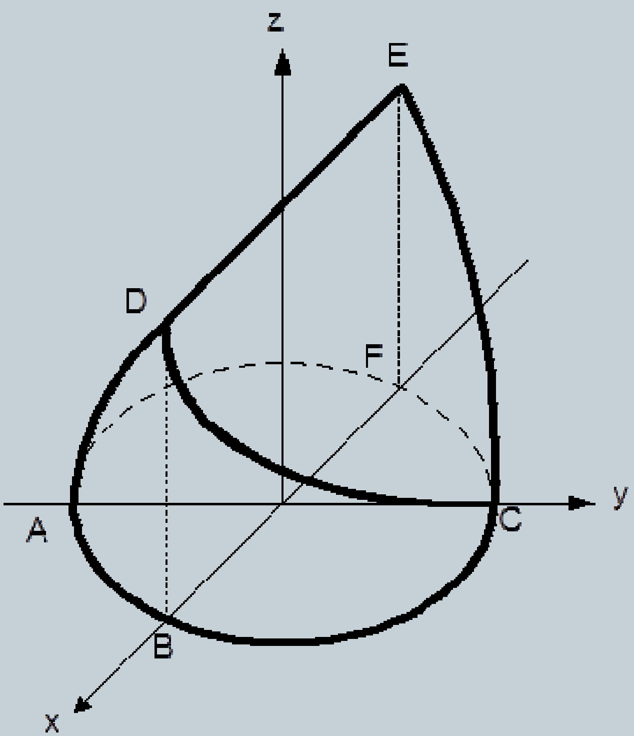
Develop a formula which describes the connection between z and the thickness t of the work piece for the test body seize d = 3 t or d = 4 t!

$$z = f(d,t)$$

Intersection body for a workshop ventilation



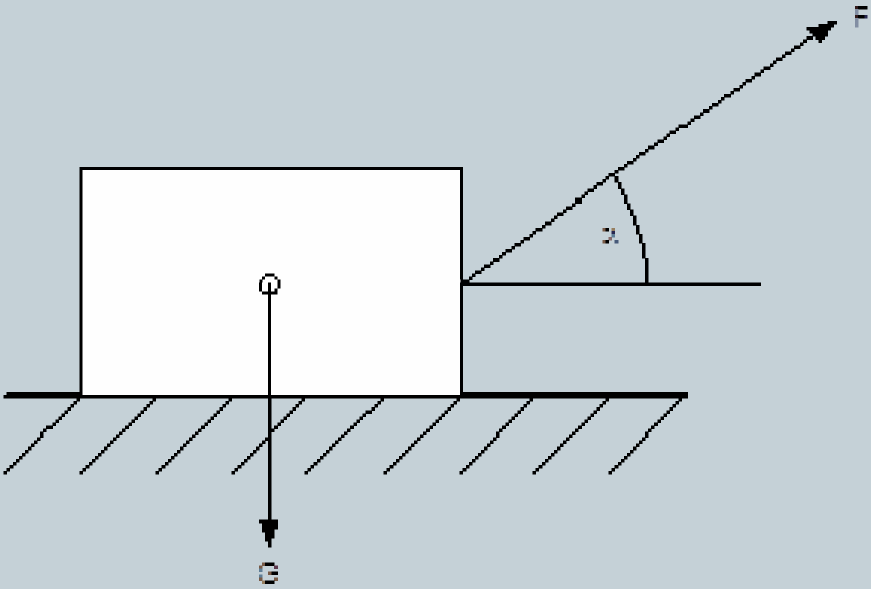
Which form must the body have to close the openings in turn?



Intersection body for a workshop ventilation

Friction – body pulled on a plane surface

A body of the mass of m is pulled on a horizontal base (frictional coefficient μ according to the selection of materials). Calculate the angle α to minimize the force F.



Friction – body pulled on a plane surface

Principle:

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0$$

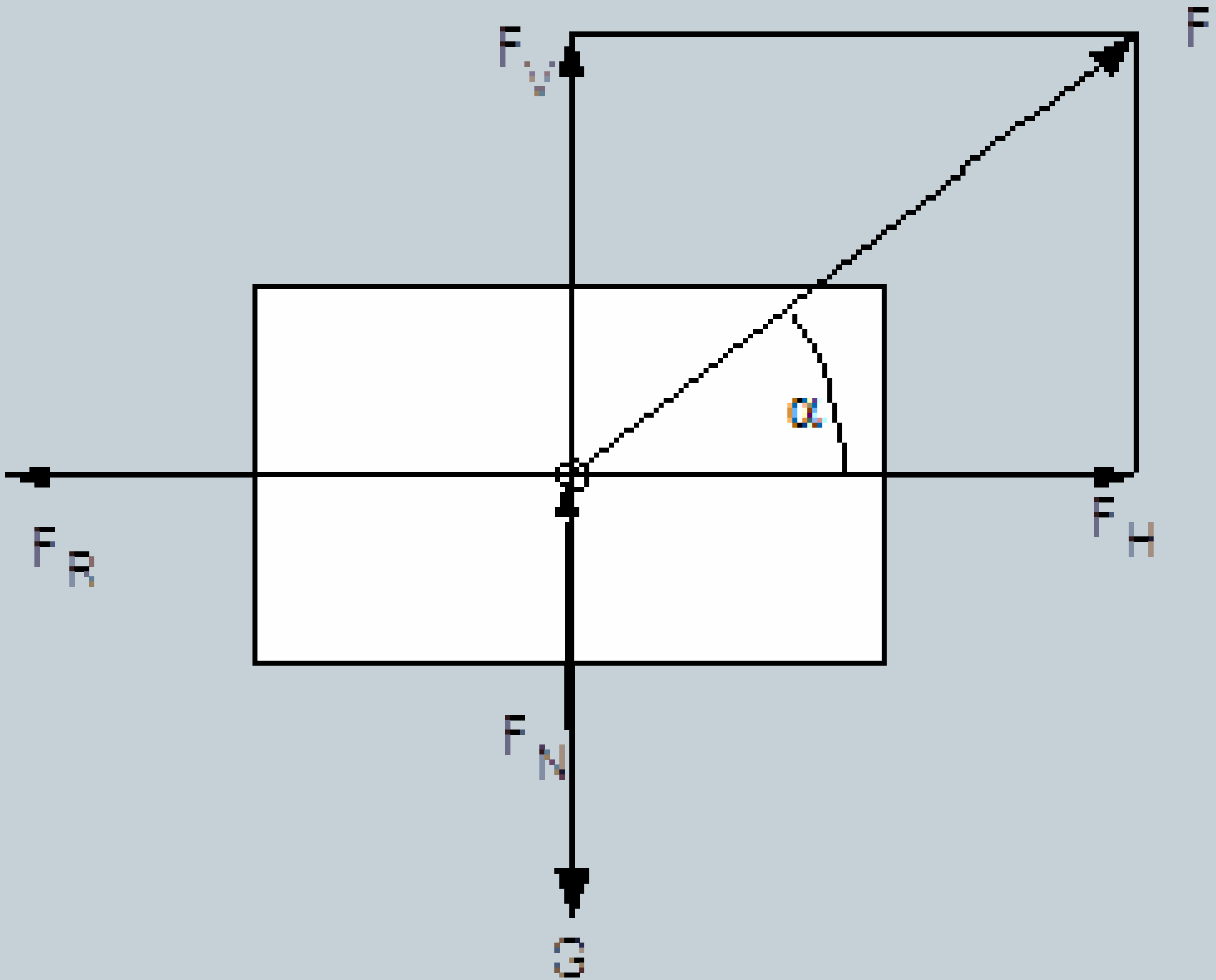
That means:

The sum of all forces in x-direction must be zero

and

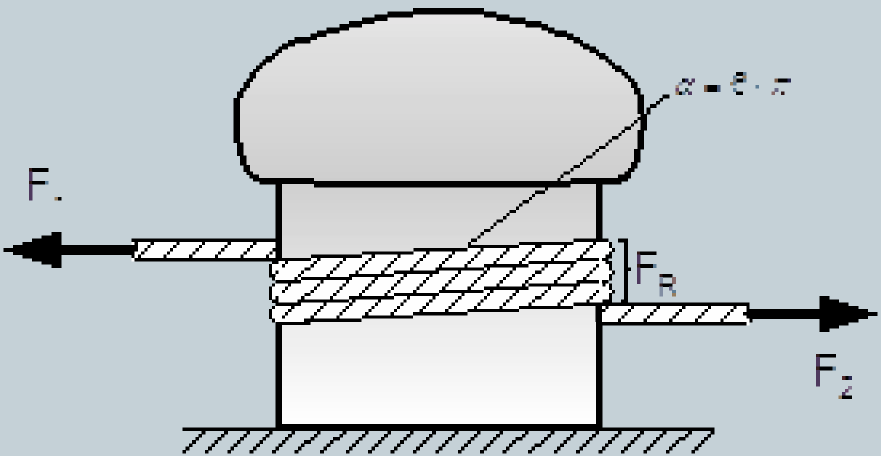
the sum of all forces in y-direction must be zero

Friction – body pulled on a plane surface



Belt friction

As an example of belt friction I use a mooring pile. It is a short steel cylinder which is used to moor ships. If you put a rope with some loops around such a mooring pile, one person is able to hold great tractive forces.



Belt friction

Measurements of an easy experiment have proved:

cylinder 1	α [rad]	0	$\pi/2$	π	2π	4π	6π
	F [N]	10	8	6	3.4	1.6	0.4
cylinder 2	α [rad]	0	$\pi/2$	π	2π	4π	
	F [N]	10	7	4.6	2.4	0.4	

Slanderousness

„Those who can do something, do it,
those who can‘t do anything, teach,
those who can‘t teach, teach teachers,
and those who can‘t teach the teachers make politics about it!“

Muriel Barbery
The elegance of the hedgehog (page 55)

Technical problems - solved by Sec 2 mathematics !

TIME 2014

Krems a. d. Donau

Settings of tasks

introduced by

Wolfgang Alvermann

BBS II Emden

July 2014

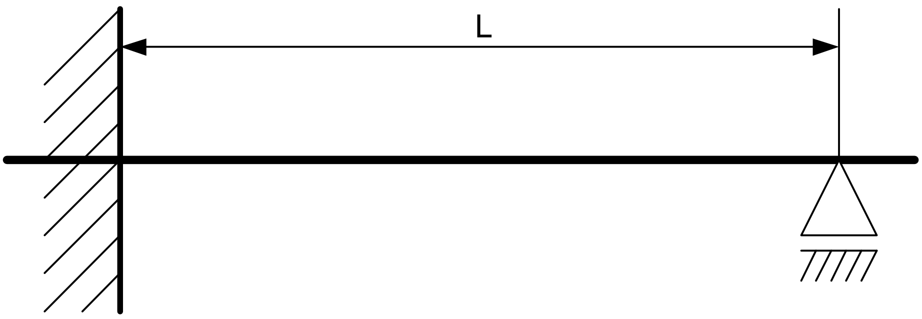
1: Deflection curves

Preliminary remark:

Deflection curves are a common problem in engineering
- roof beam
- gear shaft

Besides, there is a number of possibilities to construct a support: firm clamp, locating bearing, floating bearing.

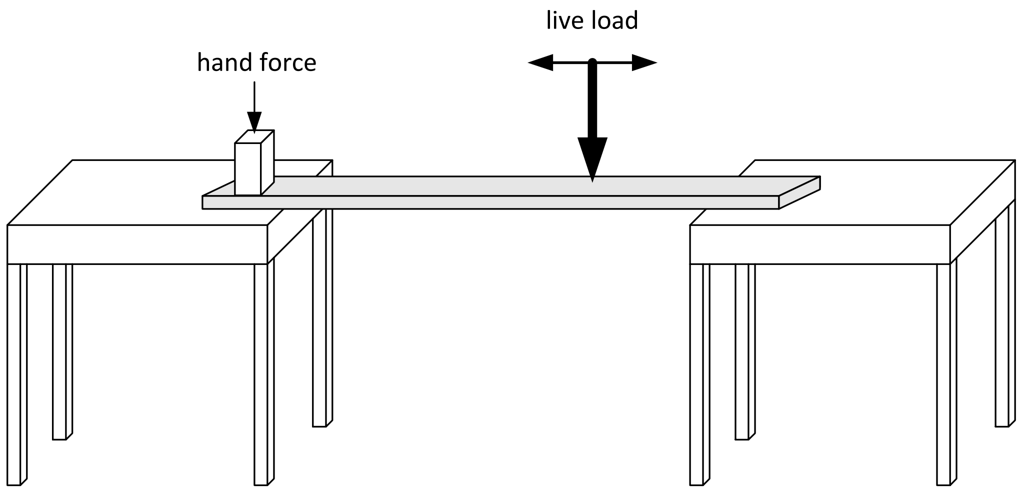
In the picture you can see a stick (beam, shaft) of the length L1 which is firmly clamped on the left side with a floating bearing on the right side. [clamp distance $L < L_1$]



The beam is stressed by a load, which can be moved freely along the beam. Its maximum sag should not be more than

$$b = \frac{1}{10} \cdot L$$

Classroom demonstration



Tasks:

a) Outline the approximate course of the deflection curve in the given drawing!

b) Find out the functional equation of the deflection curve!

c) Calculate the coordinate x_2 where the maximum sag appears!

d) Find the equation of the tangent line in the floating bearing!

e) Determine the point of inflection of the deflection curve within the internal range of [0;L]!

Deflection curves – conditions:

Sag by forces perpendicular to the beam	**Sag (deflection curve)**	$w(x)=\int w'(x)dx$	w in [m]
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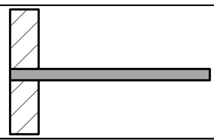
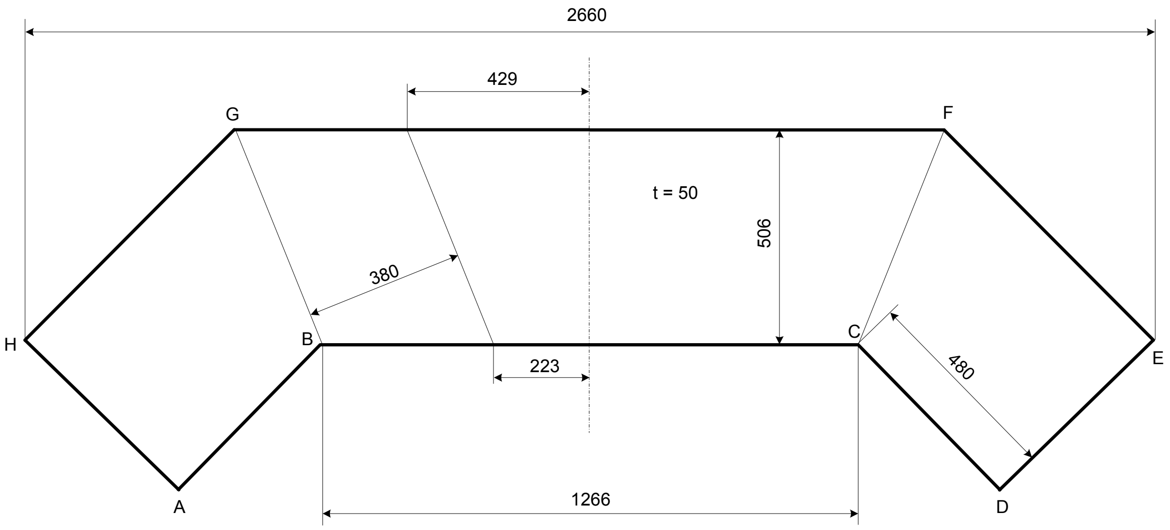
	Static moment	$w''(x)=\frac{1}{E\cdot I}\cdot(-M(x))$	M in [Nm]
	Shearing force	$w'''(x)=\frac{1}{E\cdot I}\cdot(-Q(x))$	Q in [N]
$E_{steel}=210\cdot10^9\,\frac{N}{m^2}$	**Line load**	$w''''(x)=\frac{1}{E\cdot I}\cdot q(x)$	q in $\left[\frac{N}{m}\right]$
$E_{wood}=9\cdot10^8\,\frac{N}{m^2}$		**Area moment of inertia**	I in [m^4]
		Modulus of elasticity	E in $\left[\frac{N}{m^2}\right]$
Conditions for the bearing	**Clamp**	$w(x)=0$ $w'(x)=0$	
	Bearing	$w(x_{bearing})=0$ $w''(x_{bearing})=0$	

2: A bending problem

The workshop of a metal-working company has to produce a hollow body in the form of a truncated pyramid by welding the two halves together.

A tub half has to be produced of the following strong metal with a thickness of t = 50 mm.



:

Besides, the "metal" has to be bent along the edges BG or CF in such a way that the edges from AB and BC and BC and CD stick together vertically.

If one folds the material at 90 °, you´ll get the following result:

An unsuccessful attempt costs the company 2000 € considering the size of the component and the necessary preliminary works approx.

Hence, trial-and-error is no solution.

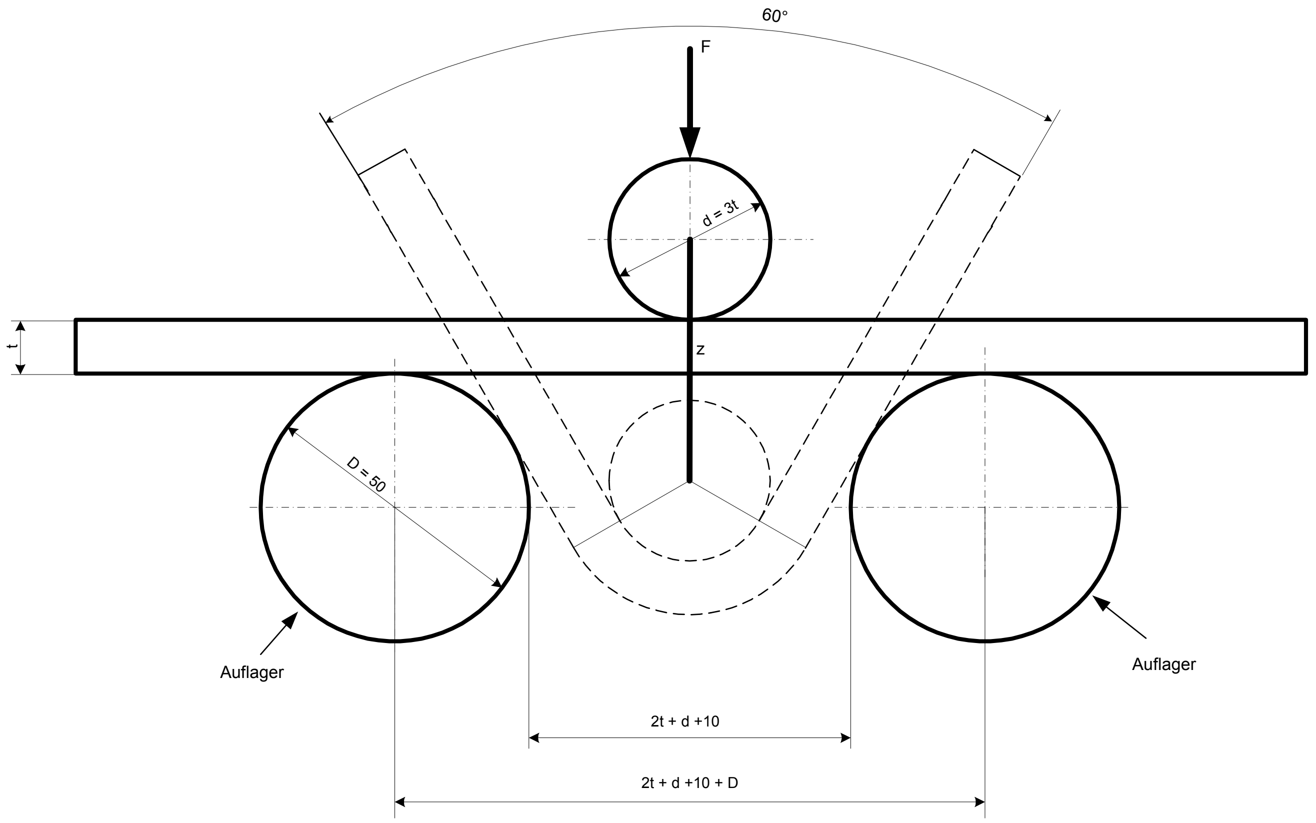
Duties:

1. Check the total length of 2660 mm of the component taking the given measures in account. Some angles must be determined in the figure for it.
2. Calculate the correct bending angle for the metal shown above!
3. Generalize the problem described above by introducing two suitable parameters and develop a tool to calculate the right bending angle for a CAS!

3: A bending-test

Preliminary remark

The Nordsee-Werke Emden (NSWE) manufacture, e.g., submarines for the federal navy. For these special ships welders engaged must pass special exams during which their work-piece is tested in a flexing test to find out if the welded joint meets the special requirements needed for building submarines.



Task

Which distance z (in mm) must the bending ram be moved to bend the material in an angle of 60 ° (see drawing)?

Develop a formula which describes the connection between z and the thickness t of the work piece for the test body seize d = 3 t or d = 4 t!

Students drawings with a CAD program have resulted in the following table:

t	2.5	5	7.5	10.0	12.5	15.0	17.5	20.0
z(3t)	31.54	36.32	40.69	45.31	49.84	54.42	58.99	63.57
z(4t)	32.45	37.94	43.43	48.97	54.41	59.93	65.39	70.83

4: A intersection body for a workshop ventilation

Preliminary remark

The workshops Emden BBS II are aired by the shown conduits.

Besides, two cylinders of the same diameter d which should be connected by soldering with each other clash.

There is in the BBS II six such ventilation ropes with three connection points in each case.

For the expense budgeting of a metal building contractor it is important to know how long the connecting lines are, around

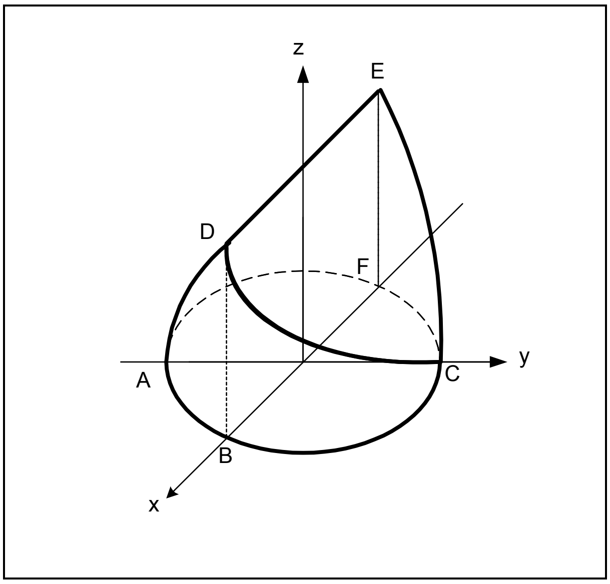
- to tear the parts before the round on a board metal (metal need).
- To be able to determine the seam length to determine the need in aid like flux, plumb line material among other things.

Task

a) Determine for the shown body with radius r and height $h = k \cdot r$ the cut curve representation in the coat winding up! $\left[k \in \mathbb{Q}^{+} \right]$

b) Determine the length of the cut curve for general r!

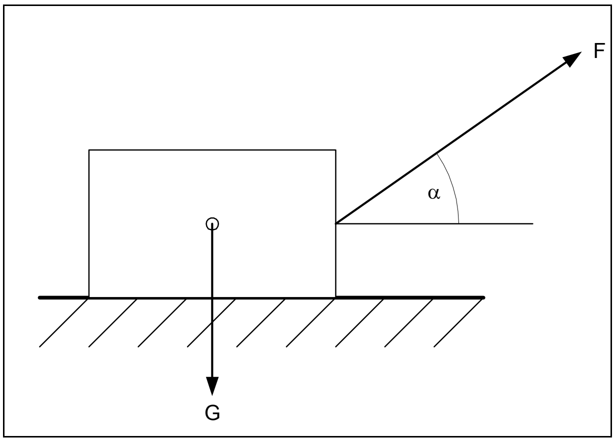
c) Calculate the coat surface!



5: Problems with friction – bodies pulled on plane surfaces

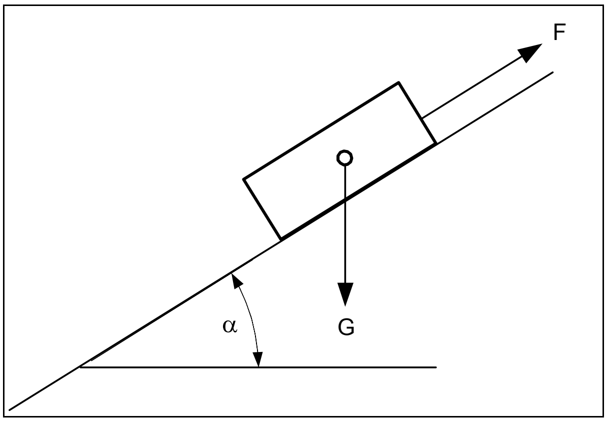
Task 1

A body of the mass of m is pulled on a horizontal base (frictional coefficient μ according to the selection of materials). Calculate the angle α to minimize the force F.



Task 2

Determine the retention force so that the body with the weight G (friction constant μ_0) does not slip off on the inclined plane!



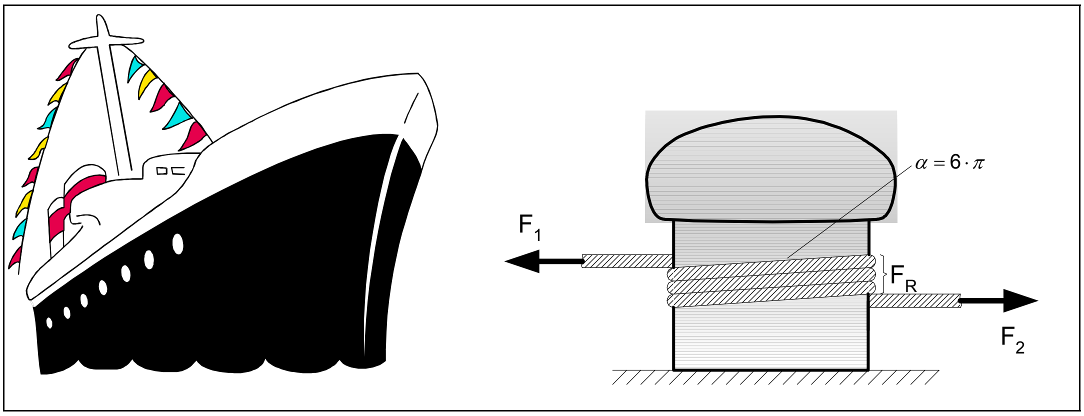
Coefficient of friction

Material	coefficient of static friction μ_0		coefficient of sliding friction μ		Rolling friction f	
	dry	lubricated	dry	lubricated		mm
Steel on steel	0,2	0,15	0,18	0,1 ... 0,08		
Steel on cast-iron	0,2	0,1	0,15	0,1 ... 0,05	Steel on steel, soft	0,5
Steel on Cu-Sn-alloy	0,2	0,1	0,1	0,06 ... 0,03		
Steel on polyamide	0,3	0,15	0,3	0,12 ... 0,05	Steel on steel, hard	0,01
Steel on friction lining	0,6	0,3	0,55	0,3 ... 0,2		
Transmission belt on cast-iron	0,5	-	-	-	Car tyre on asphalt	4,5
Roller bearing	-	-	-	0,003 ... 0,001		

6: Problem with friction no. 2 – belt friction

Preliminary remark

As an example of belt friction I use a mooring pile. It is a short steel cylinder which is used to moor ships. If you put a rope with some loops around such a mooring pile, one person is able to hold great tractive forces.



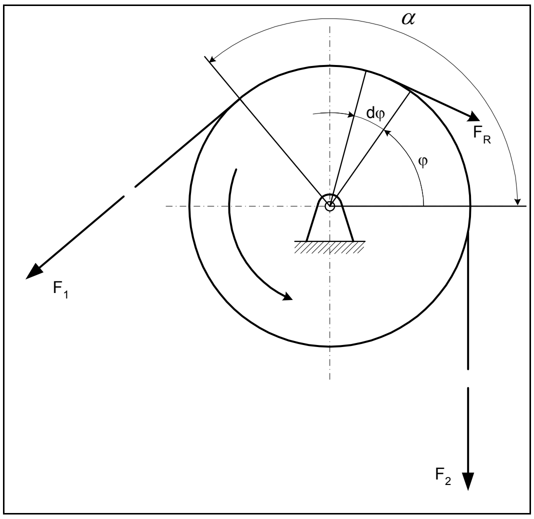
As you can see in the picture above

$$F_1 = F_2 + F_R$$

You can easily find out that the transferable tractive force becomes bigger, the greater the looping angle α is.

Moreover, the tractive force depends on the coefficient of friction, so that

$$F_1 = f(F_2, \mu, \alpha)$$



Task

Show that the law of belt friction, also called Equation of Eytelwein $F_1 = F_2 \cdot e^{\mu \cdot \alpha}$, can be found by using the measurements of a classroom experience – see the table below:

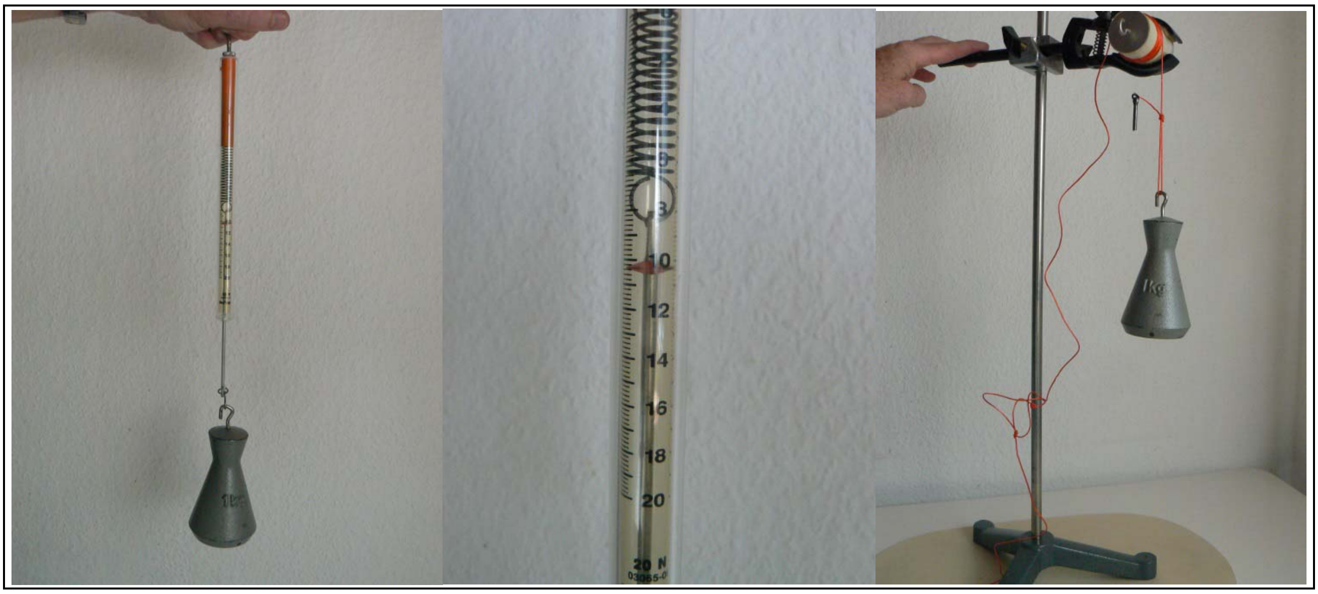
e = logarithm to the base of e

μ = coefficient of friction between rope and cylinder

α = looping angle in radian

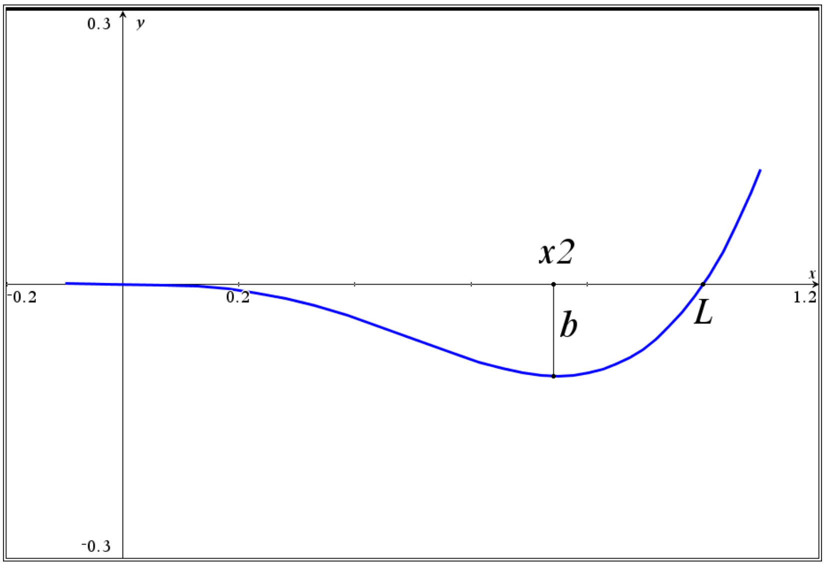
Measurements of an easy experience have proved:

cylinder 1	α [rad]	0	$\pi/2$	π	2π	4π	6π
	F [N]	10	8	6	3.4	1.6	0.4
cylinder 2	α [rad]	0	$\pi/2$	π	2π	4π	
	F [N]	10	7	4.6	2.4	0.4	



1: Deflection curves

Approach: $f: y = a_4 \cdot x^4 + a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0$



Conditions

f(0) = 0

f‘(0) = 0

f‘‘(0) = 0

It is sufficient to look upon

$f: y = a_4 \cdot x^4 + a_3 \cdot x^3$

$$\left.\begin{array}{rcl} f(L) & = & 0 \\ f(x_2) & = & -\frac{1}{10}L \\ f'(x_2) & = & 0 \end{array}\right\} \Rightarrow \begin{array}{rcl} a_3 & = & -\frac{128}{135 \cdot L^2} \\ a_4 & = & \frac{128}{135 \cdot L^3} \\ x_2 & = & \frac{3}{4} \cdot L \end{array}$$

$$\boxed{f: y = \frac{128}{135 \cdot L^3} \cdot \left(x^4 - L \cdot x^3\right) \Big| -0.2 \cdot L \le x \le 1.2 \cdot L}$$

Tangent line in the floating

Bearing: $\boxed{t: y = \frac{32}{15} \cdot (x - L)}$

Inflection point: $\boxed{W = \left(\frac{L}{2} / -\frac{8}{135} \cdot L^4\right)}$

The wright solution

Deflection curves – an incomplete worksheet

Sag by forces perpendicular to the beam	**Sag / deflection curve**	$w(x)=\int w'(x)dx$	w in [m]
	Gradient	$w'(x)=\frac{1}{E\cdot I}\cdot\int M(x)dx$	w'(x) ohne Einheit
	Static moment	$w''(x)=\frac{1}{E\cdot I}\cdot(-M(x))$	M in [Nm]
	Shearing force	$w'''(x)=\frac{1}{E\cdot I}\cdot(-Q(x))$	Q in [N]
$E_{stahl}=210\cdot 10^9\frac{N}{m^2}$	**Line load**	$w''''(x)=\frac{1}{E\cdot I}\cdot q(x)$	q in $\left[\frac{N}{m}\right]$
$E_{Holz}=9\cdot 10^8\frac{N}{m^2}$		**Area moment of inertia**	I in $[m^4]$
		Modulus of elasticity	E in $\left[\frac{N}{m^2}\right]$
	clamp	$w(x_{fest})=0$ $w'(x_{fest})=0$	
Conditions for the bearing			
	Bearing	$w(x_{Lager})=0$ $w''(x_{Lager})=0$	

Remark: A floating bearing cannot take up a static moment, so we have one other condition w‘‘(L) = 0

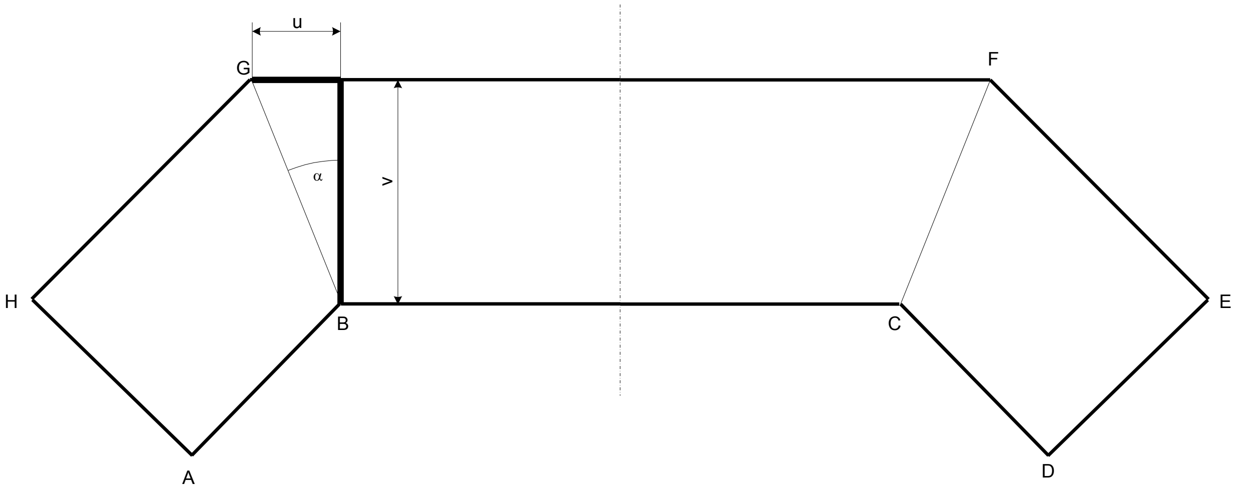
Approach: $f: y=a_4\cdot x^4+a_3\cdot x^3+a_2\cdot x^2$

Solution: $f(x)=-\frac{0.769308}{L^3}\cdot\left(x^4-2.5\cdot L\cdot x^3+1.5\cdot L^2\cdot x^2\right)$

2: A bending problem

If one folds the material at 90°, you'll get the following result:

To find a general solution, we take away all the masses and implement the parameter u, v and α.



Conditions: $\overline{AB} \perp \overline{BC}$ und $\overline{BC} \perp \overline{CD}$

$\overline{HG} \perp \overline{GF}$ und $\overline{GF} \perp \overline{FE}$

The angle α is determined by the variables u and v, the bending angle is named β.

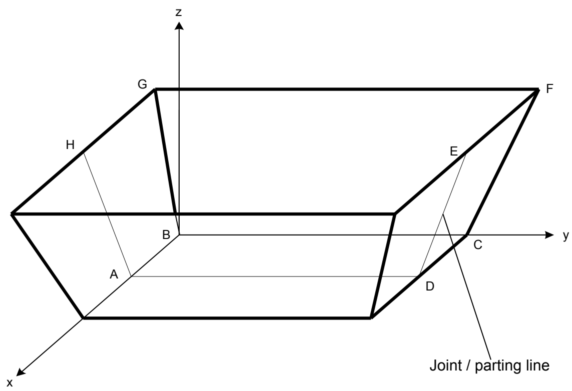
Solution

The cutting angle is the angle between two planes

E_{ABGH} und E_{BCFG}.

Formula:

$$\cos(\beta) = \left| \frac{\overrightarrow{n_1} \cdot \overrightarrow{n_2}}{\left|\overrightarrow{n_1}\right| \cdot \left|\overrightarrow{n_2}\right|} \right|$$



Therefore the coordinates of point G are to be determined using the variables u and v.

$x = -u$

$y = -u \qquad i.e. \quad G = \left(-u \,/\, -u \,/\, \sqrt{v^2 - u^2}\right)$

$z = \sqrt{v^2 - u^2}$

The normal of E_{ABGH} and E_{BCFG} are

$$\vec{n_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -u \\ -u \\ \sqrt{v^2-u^2} \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{v^2-u^2} \\ -u \end{pmatrix} \qquad \vec{n_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -u \\ -u \\ \sqrt{v^2-u^2} \end{pmatrix} = \begin{pmatrix} \sqrt{v^2-u^2} \\ 0 \\ u \end{pmatrix}$$

The dot product: $\vec{n_1} \cdot \vec{n_2} = -u^2$,

The absolute value of the product $\left|\vec{n_1}\right| \cdot \left|\vec{n_2}\right| = v^2$,

The solution $\boxed{\beta = \cos^{-1}\left(\frac{u}{v}\right)^2}$.

For u = 206 mm and v = 506 mm one receives $\beta \approx 80.4596°$

Solution with CAS (TI-Nspire)

$v1:=[1\ 0\ 0]:v2:=[0\ 1\ 0]$	$[0\ 1\ 0]$
$v3:=[-u\ -u\ \sqrt{v^2-u^2}]$	$[-u\ -u\ \sqrt{v^2-u^2}]$
⚠ $n1:=\text{crossP}(v1,v3)$	$[0\ -\sqrt{v^2-u^2}\ -u]$
⚠ $n2:=\text{crossP}(v2,v3)$	$[\sqrt{v^2-u^2}\ 0\ u]$
⚠ $\left\lvert\frac{\text{dotP}(n1,n2)}{\text{norm}(n1)\cdot\text{norm}(n2)}\right\rvert$	$\frac{u^2}{\left\lvert\sqrt{v^2-u^2}\cdot\text{conj}\left(\sqrt{v^2-u^2}\right)+u^2\right\rvert}$
$w1(u,v):=\cos^{-1}\left(\frac{u^2}{\left\lvert\sqrt{v^2-u^2}\cdot\text{conj}\left(\sqrt{v^2-u^2}\right)+u^2\right\rvert}\right)$	*Done*
$(w1(206,506))$▸DD	80.4596°
© The "hand–made"–solution	
$w2(u,v):=\cos^{-1}\left(\frac{u^2}{v^2}\right)$	*Done*
$(w2(206,506))$▸DD	80.4596°

3: A three-point bending test

It is very difficult to find a solution analytically.
Students drawings with a CAD-program have resulted in the following table:

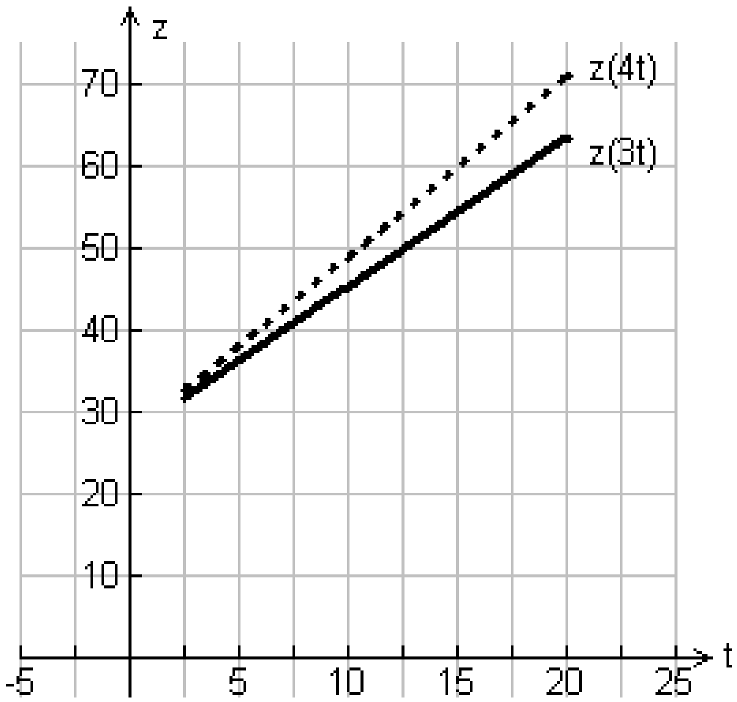
t	2.5	5	7.5	10.0	12.5	15.0	17.5	20.0
z(3t)	31.54	36.32	40.69	45.31	49.84	54.42	58.99	63.57
z(4t)	32.45	37.94	43.43	48.97	54.41	59.93	65.39	70.83

Using linear regression, you receive the following functions:

$z(3t) = 1.825 \cdot t + 27$ for $t \geq 2.5$

$z(4t) = 2.196 \cdot t + 27$ for $t \geq 2.5$

Diagram



To prove the result of the linear regression, you can also think of this task as an extremal problem

Declare two lists: t → lx

z(3t) → ly

and a function $g(x) = m \cdot x + b$

The solution with a CAS

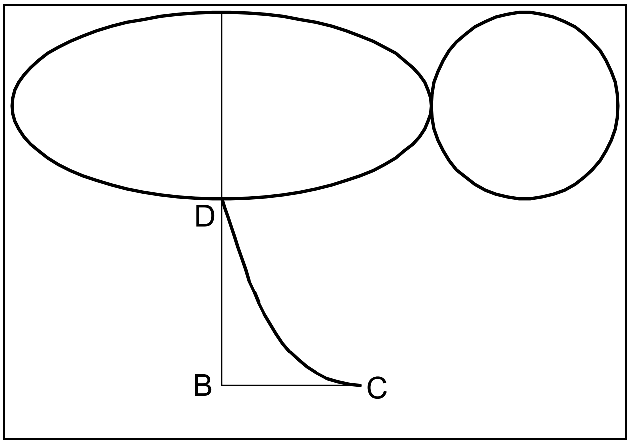
$lx:=\{2.5,5,7.5,10,12.5,15,17.5,20\}$	$\{2.5,5,7.5,10,12.5,15,17.5,20\}$
$ly:=\{31.54,36.32,40.69,45.31,49.84,54.42,58.99,63.57\}$	$\{31.54,36.32,40.69,45.31,49.84,54.42,58.99,63.57\}$
$g(x):=m\cdot x+b$	*Done*
$\text{sum}\left((g(lx-ly))^2\right)\rightarrow d(m,b):d(m,b)$	$8\cdot b^2-581.36\cdot b\cdot m+10740.6\cdot m^2$
$\text{sum}\left((g(lx)-ly)^2\right)\rightarrow d(m,b):d(m,b)$	$8\cdot b^2+b\cdot(180.\cdot m-761.36)+1275.\cdot m^2-9523.5\cdot m+18989.1$
$\frac{d}{dm}(d(m,b))=0\rightarrow eq1$	$2550.\cdot m+180.\cdot b-9523.5=0$
$\frac{d}{db}(d(m,b))=0\rightarrow eq2$	$16.\cdot b+180.\cdot m-761.36=0$
solve($eq1$ and $eq2$,m,b)	$m=1.82514$ and $b=27.0521$
$g(x)\vert m=1.8251428571429$ and $b=27.052142857143$	$1.82514\cdot x+27.0521$
$f1(x):=1.8251428571429\cdot x+27.052142857143$	*Done*

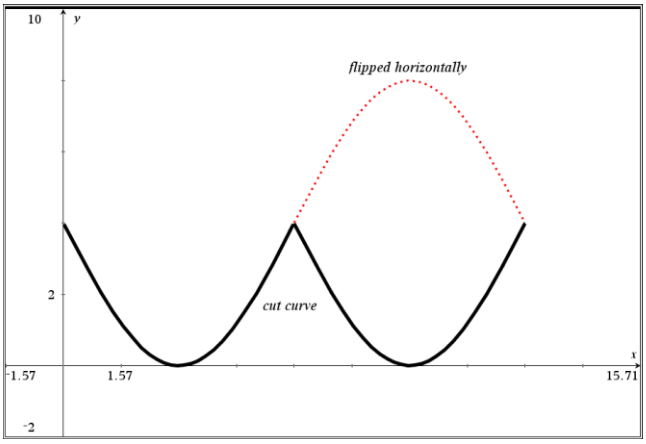
4: A intersection body for a workshop ventilation

Equation for the line of intersection (for h = 2r)

The left picture shows a part DC of the line of intersection from the unrolling.

The right picture shows the whole line of intersection with a horizontally flipped part.





Conditions for the cut curve:

$f(0) = 2 \cdot r \qquad f'(0) = -2$

$f\left(\frac{\pi}{2} \cdot r\right) = 0 \qquad f'\left(\frac{\pi}{2} \cdot r\right) = 0$

Set up: $f(x) = a \cdot \sin\left[b\left(x + c\right)\right] + d$

One can read out directly: d = 2r   a = -2r   c = 0

For b one finds out $b = \frac{1}{r}$

$$\boxed{f(x) = 2r \cdot \left[1 - \sin\left(\frac{x}{r}\right)\right] \quad \text{für } x \in \left[0; \frac{\pi}{2} \cdot r\right]}$$

$f' = -2 \cdot \cos\left(\frac{x}{r}\right)$

The length of the line of intersection

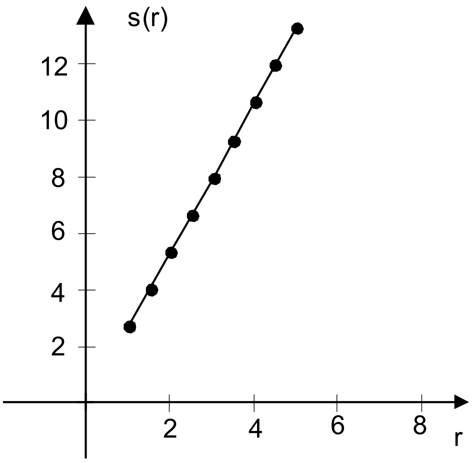
Formula: $L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$

In this case: $s(r) = \int_0^{\frac{\pi}{2}\cdot r} \sqrt{1+\left[-2\cos\left(\frac{x}{r}\right)\right]^2}\,dx$

CAS are not able to solve such a definite integral, but if one declares r = 1, r = 2, … you'll get the following table for the length of DC.

r	s(r)
1	2.63518
1.5	3.95277
2	5.27036
2.5	6.58795
3	7.90555
3.5	9.22314
4	10.5407
4.5	11.8583
5	13.1759

$\Rightarrow$ $s(r) \approx 2.63518 \cdot r$



To verify this solution, you can also determine the length of an ellipse with the half-axis $a = \sqrt{5}$ and $b = r$.

Total area of the corpus

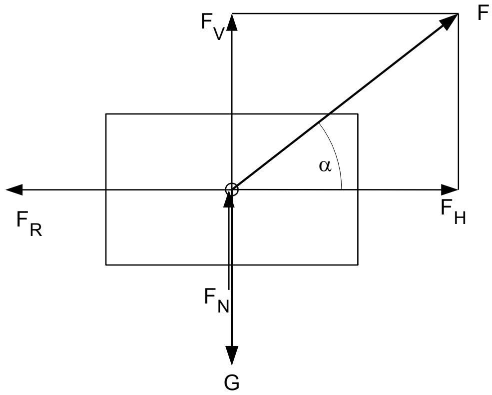
- Ellipse with $a = \sqrt{5} \cdot r$ und $b = r$
- Circle with R = r and
- $A = 4 \cdot \int_0^{\frac{\pi}{2}\cdot r} f(x)dx$

$$A_{ges} = \pi \cdot \sqrt{5}r \cdot r + \pi \cdot r^2 + 4 \cdot (\pi - 2) \cdot r^2$$
$$A_{ges} \approx 11.308 \cdot r^2$$

5: Problem with friction – bodies pulled on plane surfaces

A horizontal plane

1. Take away the bottom support



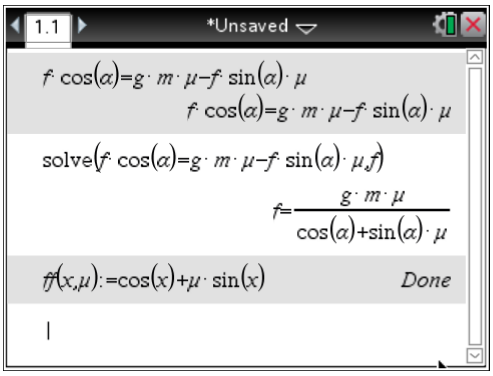
2. Connection between the forces

$F_H = F_R \qquad F_R = \mu \cdot F_N$

$F_N = G - F_V; \quad G = m \cdot g;$

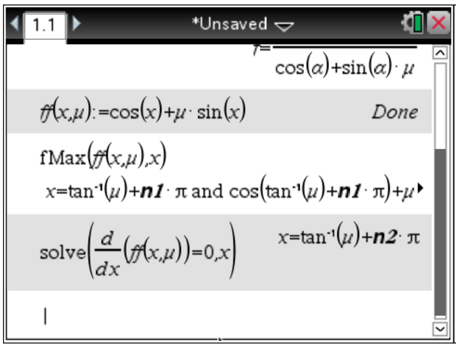
$F_V = F \cdot \sin(\alpha) \Rightarrow F_N = m \cdot g - F \cdot \sin(\alpha)$

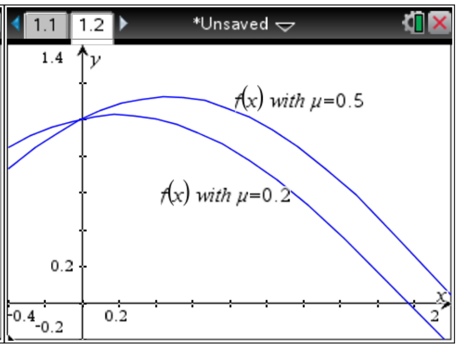
$F_H = F \cdot \cos(\alpha)$



3. To determine the minimum force

It is enough to determine the maximumof the denominator to find the lowest force.





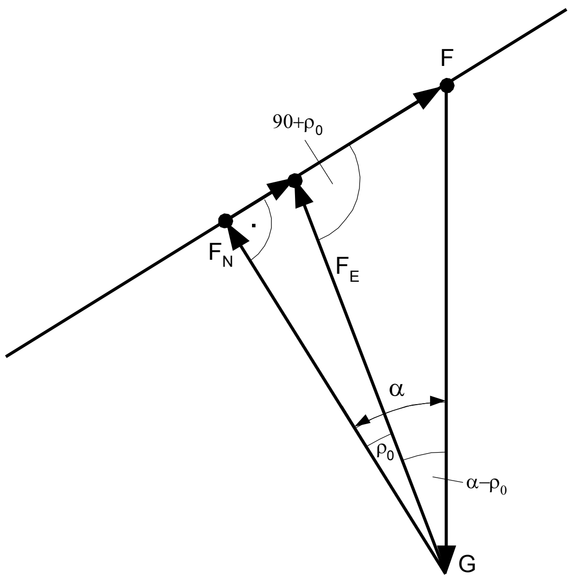
General formula:

$$\alpha = \tan^{-1}(\mu)$$
$$\text{für } \mu = 0.2 \Rightarrow \alpha = 11.3°$$

This formula is part of textbooks of mechanical engineering.

A sloped plane

1. Take away the bottom support



2. Connection between the forces

$F_{Ro:}$ Force of static friction

$F_{N:}$ Normal force

$F_E = F_{Ro} + F_N$

This way we build an oblique triangle; with the law of sine we get

$$\frac{F}{\sin(\alpha-\rho_0)} = \frac{G}{\sin(90+\rho_0)}$$

$$F = G\cdot\frac{\sin(\alpha-\rho_0)}{\sin(90+\rho_0)}$$

$$\boxed{F = G\cdot\frac{\sin(\alpha-\rho_0)}{\cos(\rho_0)}}$$

Addition theorems:

$$F = G\cdot\frac{\sin(\alpha)\cdot\cos(\rho_0)-\cos(\alpha)\cdot\sin(\rho_0)}{\cos(\rho_0)}$$

$$F = G\cdot\left[\sin(\alpha)-\frac{\sin(\rho_0)}{\cos(\rho_0)}\cdot\cos(\alpha)\right]$$

with $\mu_0=\tan(\rho_0)=\frac{\sin(\rho_0)}{\cos(\rho_0)}$

$$\boxed{F=G\cdot\left[\sin(\alpha)-\mu_0\cdot\cos(\alpha)\right]}$$

There are two cases which can be distinguished:

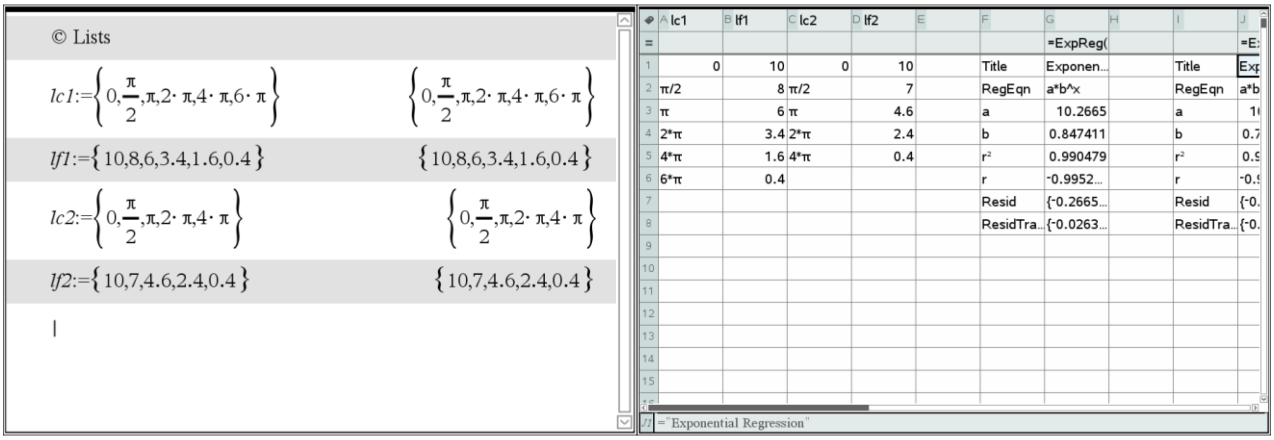
$\underline{\rho_0=\alpha}$ that means: $F=G\cdot\frac{\sin(0)}{\cos(\rho_0)}=0$

$\underline{\rho_0=0}$ that means: $F=G\cdot\frac{\sin(\alpha)}{\cos(0)}=G\cdot\sin(\alpha)$

6: Belt friction

Derivation from measured data

a) Evaluation

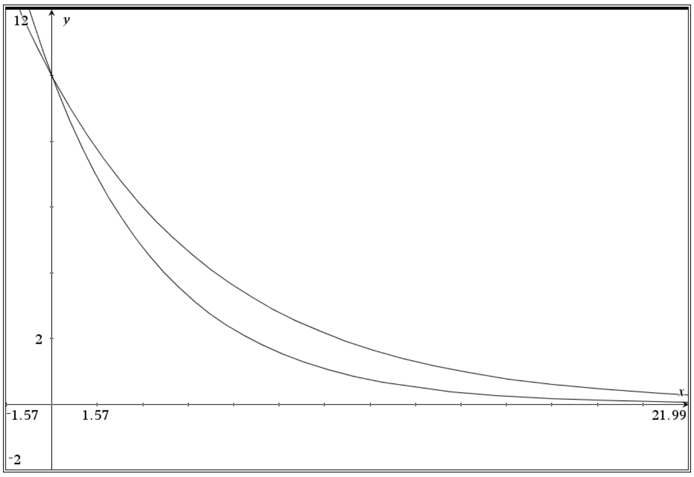


	A lc1	B lf1	C lc2	D lf2	E	F	G	H	I	J
=							=ExpReg(			=E:
1	0	10	0	10		Title	Exponen...		Title	Exp
2	π/2	8	π/2	7		RegEqn	a*b^x		RegEqn	a*b
3	π	6	π	4.6		a	10.2665		a	1(
4	2*π	3.4	2*π	2.4		b	0.847411		b	0.7
5	4*π	1.6	4*π	0.4		r²	0.990479		r²	0.9
6	6*π	0.4				r	-0.9952...		r	-0.
7						Resid	{-0.2665...		Resid	{-0.
8						ResidTra...	{-0.0263...		ResidTra...	{-0.

With exponential regression one finds

$f1(x) \approx 10.2665 \cdot 0.8474^x \longrightarrow f3(x) = 10 \cdot 0.85^x$

$f2(x) \approx 10.4763 \cdot 0.7748^x \longrightarrow f4(x) = 10 \cdot 0.79^x$



b) The differences

With $a = e^{\ln(a)}$ follows

$f3(x) = 10 \cdot e^{-0.1625 \cdot x} = \dfrac{10}{e^{0.1625 \cdot x}}$ und

$f4(x) = 10 \cdot e^{-0.2357 \cdot x} = \dfrac{10}{e^{0.2357 \cdot x}}$.

X: wrap-around angle;

Still the values 0.1625 or 0.2357 are to be explained.

There friction is different between both cylinders and the rope

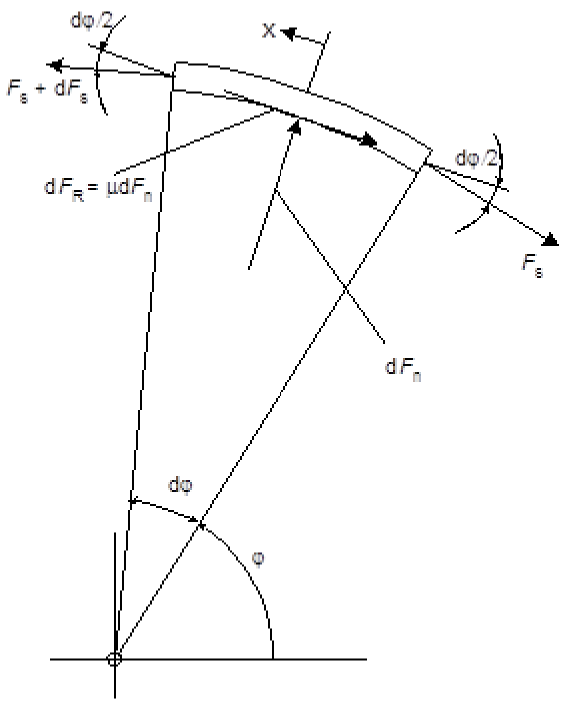
The coefficient of friction μ is 0.1625 respectively 0.2357.

The general formula for belt friction $\boxed{F_1 = F_2 \cdot e^{\mu \cdot \alpha}}$

- F_1 the force to hold
- F_2 the force one needs to hold F_1
- α the wrap-around angle
- μ coefficient of friction

Analytical deduction

If you look at the small sector of the circle (see graphic) one can determine the balance of forces.



$$(F_s + \mathrm{d}F_s)\cdot\cos(\mathrm{d}\varphi/2) - \mu\cdot\mathrm{d}F_n - F_s\cdot\cos(\mathrm{d}\varphi/2) = 0 \quad (1)$$

$$\mathrm{d}F_n - (F_s + \mathrm{d}F_s)\cdot\sin(\mathrm{d}\varphi/2) - F_s\cdot\sin(\mathrm{d}\varphi/2) = 0 \quad (2)$$

For a very small angle one can write:

$$\cos(\mathrm{d}\varphi/2) = 1 \text{ and } \sin(\mathrm{d}\varphi/2) = \mathrm{d}\varphi/2 \quad (3)$$

You can also ignore the higher-order product $\mathrm{d}F_s\cdot\mathrm{d}\varphi/2$

The result of (1) and (2)

$$\left.\begin{aligned}\mathrm{d}F_s &= \mu\cdot\mathrm{d}F_n = \mathrm{d}F_R\\ \mathrm{d}F_n &= F_s\mathrm{d}\varphi\end{aligned}\right\}\Rightarrow \mathrm{d}F_s = \mu\cdot F_s\mathrm{d}\varphi$$

With $\mu = \mu_0 = const.$ and $\varphi = \alpha$ you get for the state of equilibrium the differential equation

$$\boxed{\mathrm{d}F_s = \mu_0\cdot F_s\mathrm{d}\alpha}$$

Solve this DE with "separation of the variables"

$$\frac{\mathrm{d}F_s}{F_s} = \mu_0\cdot\mathrm{d}\alpha$$

$$\int\frac{\mathrm{d}F_s}{F_s} = \int\mu_0\cdot\mathrm{d}\alpha \quad\Rightarrow\quad \ln(F_s) = \mu_0\cdot\alpha + \ln(C)$$

$$\ln(F_s) - \ln(C) = \ln\left(\frac{F_s}{C}\right) = \mu_0\cdot\alpha \qquad\qquad F_s = C\cdot e^{\mu_0\cdot\alpha}$$

With $F_s = F_1$ and the initial value $C = F_s(0) = F_2$ you'll get the *Law of Eytelwein*

$$\boxed{F_1 = F_2\cdot e^{\mu_0\cdot\alpha}}$$