Students' Comparison of Their Trigonometric Answers with the Answers of a Computer Algebra System in Terms of Equivalence and Correctness

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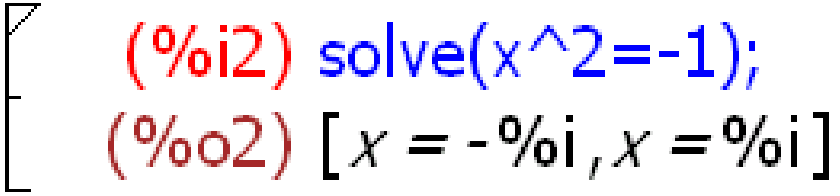
Answers offered by CAS

- could be evaluated from different points of view
  - professional user's
    - mathematically correct
    - somewhat flexible output allowed (nuances do not confuse so much)
      - radians/degrees
    - …
  - student's
    - according to school mathematics
    - nuances could be important
    - …
  - …

=?



#	PROBLEM	Ax	De	Mc	Mp	Mm	Mu	Re
L6	derivative of above is 0 & above at 0 is 0	#	#	$\star^{17}$	$\star$	○	#	●
L7	$\sqrt{4r+4\sqrt{r+1}}$		○	$\star$	●	○	●	●
L8	$\sqrt{r}$	○				○	●	
L9	$\frac{z\pi}{2} - \pi^z\zeta(1-z) \Rightarrow 0$	—				●		
	M. Equations							
M1		●		●	●	○	●	●
M2	solve$(3x^3 - 18x^2 + 33x - 19 = 0, \mathbf{R})$	○	●	$\star$	$\star$	○	$\star$	●
M3	solve$(x^4 + x^3 + x^2 + x + 1 = 0)$	♮	●	●	●	○	●	●
M4	verify a solution of the above	●	●	●	●	●	$\star$	●
M5	solve$(x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23)$							
M6	solve$(x^7 - 1 = 0) \Rightarrow x = \{1, \{e^{\pm 2k\pi i/7}\}_{k=1}^{3}\}$	♮	$\star$	●	●	$\star$	♮	♮
M7	solve$(x^8 - 8x^7 + \cdots - 140x + 46 = 0)$			●	●	●	●	●
M8	solve$(e^{2x} + 2e^x + 1 = z, x)$							
M9	solve$(e^{2-x^2} = e^{-x}) \Rightarrow x = \{-1, 2\}[+$							
M10	solve$(e^x = x) \Rightarrow x = -W_n(-1)$ $(n$							
M11	solve$(x^x = x) \Rightarrow x = \{-1, 1\}$							

Michael Wester. *Computer Algebra Systems. A Practical Guide*. 1999

– 542 problems

- ● success! (hurrah!)
- # incompletely simplified, but some useful transformations were performed (groan)
- ε a surprising error occurred (ack!)
- ….

The unexpected answers

confusing and obstructive

or

opportunities (Paul Drijvers) and

a catalyst for rich mathematical discussion (Robyn Pierce, Kaye Stacey)

Opportunity, catalyst

Drijvers, P. (2002)

> *The first obstacle is: The difference between the algebraic representations provided by the CAS and those students expect and conceive as 'simple'. ... Recognizing equivalent expressions is a central issue in algebra, and still is when working in a computer algebra environment.*

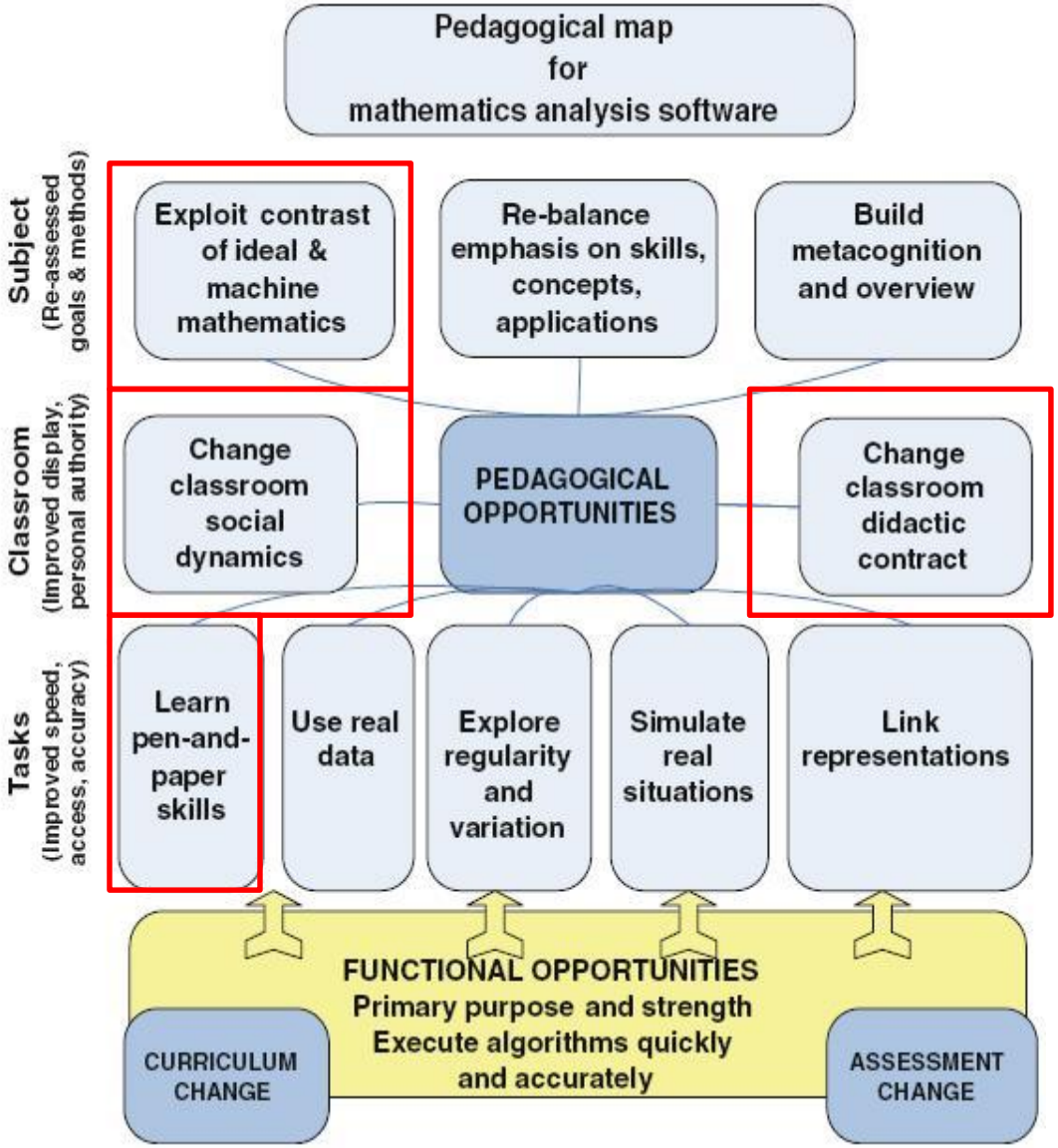
Pierce, R., & Stacey, K. (2010)

> *Unexpected mathematical results may be distracting and disheartening, but they are also pedagogical opportunities since they be used to provoke rich mathematical discussion*.

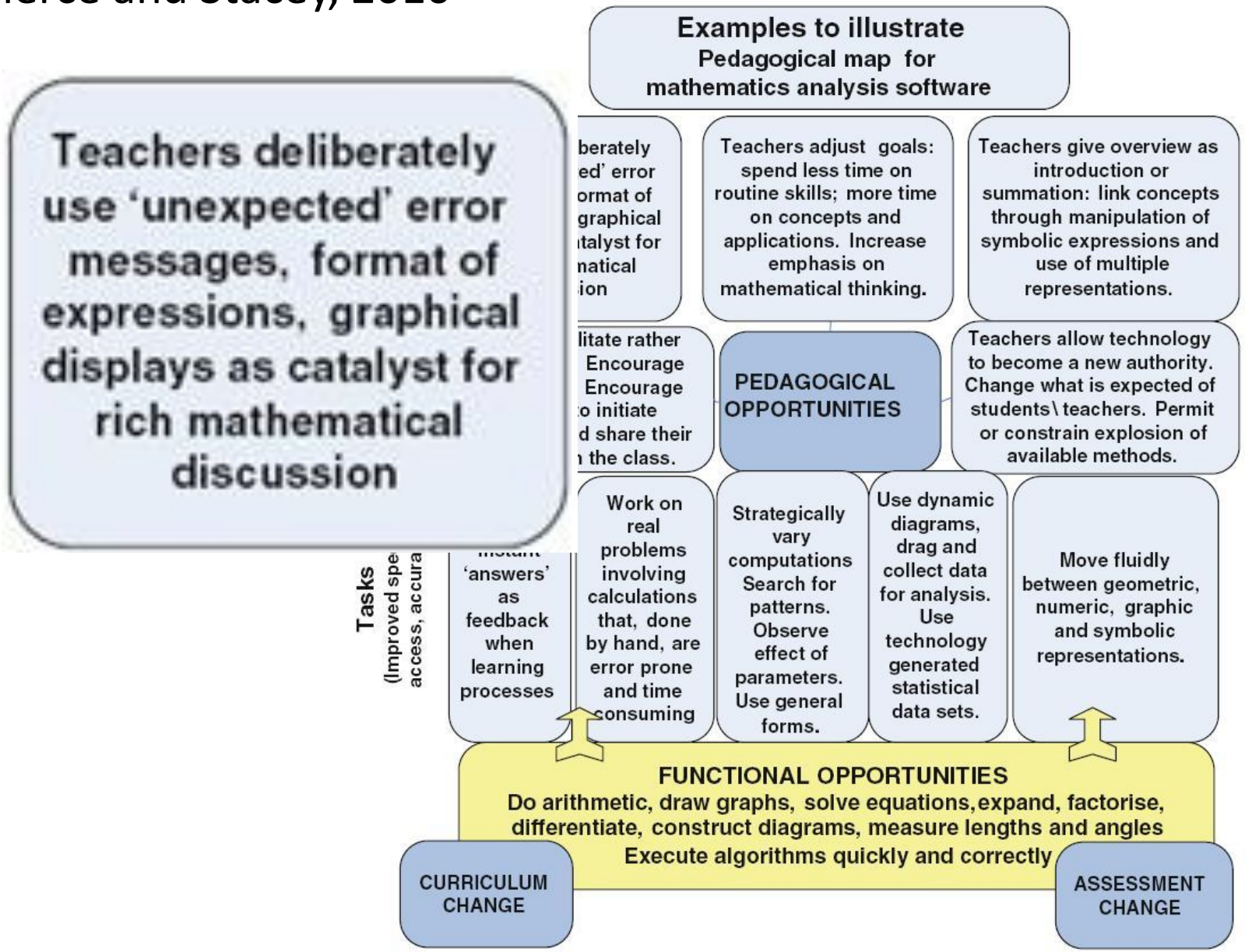
Buteau, C., Marshall, N., Jarvis, D. H., & Lavicza, Z. (2010)

> *Although practitioners have to deal with unusual or unexpected behaviour of CAS, this was occasionally shown to provide pedagogical opportunities*.

Pierce and Stacey, Mapping pedagogical opportunities provided by mathematics analysis software. *International Journal of Computers for Mathematical Learning*, *15*(1), 1-20. 2010

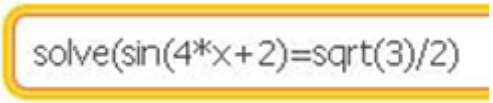


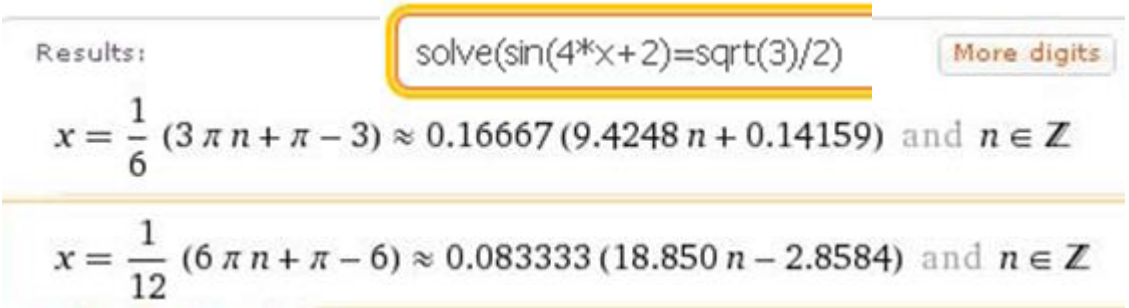
Pierce and Stacey, 2010



Student and CAS

- Student solves with CAS.
- One-step
  - equation → answer





- Students' Comparison of Their Trigonometric Answers with the Answers of a Computer Algebra System in Terms of Equivalence and Correctness

A student is charged with the task of comparing the answers

- What will happen when students themselves are encouraged to analyze differences, equivalence and correctness of their own answers and CAS answers?
- What differences do they notice foremost?
- How do they understand correctness of the answers?
- Are students able to ascertain equivalence/non-equivalence?
- How do they explain equivalence/non-equivalence?
- Are there any differences in this regard between different types of equations and answers?

Lessons

- First-year university students
- Course ”Elementary mathematics”
  - a somewhat repetitious course of school mathematics
- 90 minutes
    - an introduction
      - an overview of the lesson, the aims of the study
    - a period of equation-solving (ca 70 minutes)
    - closing (saving and copying data)
  - were taught by the first author (not a regular teacher of the course)

- Students in pairs (discussion!!!)
  - discussions were audio-taped
- The students
  - first solved an trigonometric equation (correctly or not) without CAS
  - then with a particular CAS
  - analyzed differences, equivalence and correctness of their own answers and CAS answers

Trigonometric equation

- Solving trigonometric equations
  - the variety of possible presentations of solutions
  - units of measurement
  - general and particular solutions

- Variety of their answers
  - several reasonable representations of the correct answer
  - different solution strategies, different-looking but still equivalent answers
  - different formulae in different regions
    - solution for $\sin x = m$

$$x = \arcsin m + 2n\pi,\ n \in \mathbb{Z}$$
$$x = \pi - \arcsin m + 2n\pi,\ n \in \mathbb{Z}$$

    - or (as in Estonian textbooks, for example)

$$x = (-1)^n \arcsin m + n\pi,\ n \in \mathbb{Z}$$

iJMEST
International Journal of Mathematical Education in Science and Technology

How to 'check the result'? Discourse revisited

SERGEI ABRAMOVICH
School of Education and Professional Studies,

$$2 + \cos^2 2x = (2 - \sin^2 x)^2$$

Alan completed his solution with the formula

$$x = \pm \arcsin \frac{1}{\sqrt[4]{3}} + \pi n$$

ed the square root from both sides of equation (3), Betsy completed the formula

$$x = \pm \arccos\left(\pm\sqrt{\frac{3-\sqrt{3}}{3}}\right) + 2\pi n \qquad (4)$$

Christina's solution.

$$x = \pm\frac{1}{2}\arccos\frac{3-2\sqrt{3}}{3} + \pi n$$

Dave

$$x = \pm \arctan\sqrt{\frac{1+\sqrt{3}}{2}} + \pi n$$

Order

- The students had worksheets with equations and tasks
- The order of solvable equations
  - prescribed
- The students
  - first solved an equation (correctly or not) without a CAS
  - then with a particular CAS
- WolframAlpha (in the first three equations)
- A specific CAS was prescribed for the equation
  - the expected difference between the students' answers and the CAS answer
  - initiate an "intrigue", the effect of different representations

- Solve an equation $\sin(4x+2) = \frac{\sqrt{3}}{2}$ (without the computer at first).
  - How confident are you in the correctness of your answer?
- Solve the equation with the CAS WolframAlpha using the command solve.

  - How unexpected is the CAS answer at first view?
  - Analyze the accordance of your answer with the CAS answer! If you want to complement/correct your solution, please use the green pen.
  - What are the differences between your answer and the CAS answer?
  - How are your answer and the CAS answer related (analyze equivalence/nonequivalence, particular solutions/general solutions)?
  - Rate the correctness of your (possibly corrected) answer.
  - Rate the correctness of the CAS answer.
  - Rate the equivalence/non-equivalence of your (possibly corrected) and CAS answers.

$$\sin(4x + 2) = \frac{\sqrt{3}}{2}$$

$$2\sin 2x \cos 2x + \cos 2x = 0$$

in the interval [-30°; 0]

$$\frac{\tan^2 x}{\tan x} = 0$$

$$\tan^3 x = \tan x$$

$$\tan\left(x + \frac{\pi}{4}\right) = 2\cot x - 1$$

$$2\cos^2 x + 4\cos x = 3\sin^2 x$$

$$1 - \cos x = \sqrt{3}\sin x$$

Three equations

- 112 instances of equation-solving (38 pairs of students)
- The student worksheets and audio-tapes (in questionnable places) were analyzed
- For each equation in paper
  - how students solved the equation (common mistakes)
  - correctness of the students' answer / the students' confidence in their answer
  - How unexpected is the CAS answer at first view?
  - correctness of the students' answer / the students' opinion about correctness of their answer
  - correctness of the students' answer / the students' opinion about correctness of CAS answer
  - equivalence of the students' and CAS answer / the students' opinion about equivalence of their and CAS answer
  - What are the differences between your answer and the CAS answer?
  - How are your answer and the CAS answer related (analyse equivalence/non-equivalence, particular solutions/general solutions)?

Results - students and questionnaires

- 85 students have filled the pre- and post-questionnaire
- The students, on average, rated their level of school mathematics knowledge at 4.07 (on scale 1-5)
- The students had very limited experiences with CAS
  - WolframAlpha was the most familiar one
- Work in pairs is not very common in Estonian schools and the university
- Students like to work in pairs
- Post-questionnaire – focus on work in pairs

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

- The first equation was quite complicated for the students
- 12 pairs solved it correctly

$$x=(-1)^n\frac{\pi}{12}-\frac{1}{2}+\frac{n\pi}{4},n\in Z$$

- 21 pairs solved it incorrectly
  - The most common mistakes were connected to measure units - radians and degrees

$$x=15^\circ+45^\circ n-\frac{1}{2}$$

$$x=14.5^\circ+45^\circ n$$

- 5 pairs provided particular solution while the general solution was needed

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Correctness of the students' answer / the students' confidence in their answer

	Stud. very confident	Stud. quite confident	Stud. do not know	Stud. quite unsure	Stud. very unsure	
Math. correct	**2**	**6**	**3**			**11**
Math. incorrect	**9**	**2**	**1**	**3**	**5**	**20**
Math. particular/general	**1**	**2**	**2**			**5**
	12	**10**	**6**	**3**	**5**	

- 9 pairs, which were very confident in their answer, had actually a wrong answer
  - 7 of them marked their answer as correct even after solving the equation with CAS

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Correctness of the students' answer / the students' confidence in their answer

	Stud. very confident	Stud. quite confident	Stud. do not know	Stud. quite unsure	Stud. very unsure	
Math. correct	2	6	3			11
Math. incorrect	9	2	1	3	5	20
Math. particular/general	1	2	2			5
	12	10	6	3	5	

- 3 pairs with correct answer did not know how confident or unsure they were
  - 2 of them marked their answer as wrong after seeing the CAS answer

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Correctness of the students' answer / the students' confidence in their answer

	Stud. very confident	Stud. quite confident	Stud. do not know	Stud. quite unsure	Stud. very unsure	
Math. correct	**2**	**6**	**3**			**11**
Math. incorrect	**9**	**2**	**1**	**3**	**5**	**20**
Math. particular/general	**1**	**2**	**2**			**5**
	12	**10**	**6**	**3**	**5**	

- There was no intrigue with pairs who were not confident
  - They had wrong answer and had marked their answer as wrong after häving seen the CAS answer

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

- How unexpected is the CAS answer at first view?

school answer

$$x=(-1)^n\frac{\pi}{12}-\frac{1}{2}+\frac{n\pi}{4},n\in Z$$

CAS answer

$$x=\frac{1}{6}(3\pi n+\pi-3)\approx 0.16667(9.4248n+0.14159) \text{ and } n\in\mathbb{Z}$$

$$x=\frac{1}{12}(6\pi n+\pi-6)\approx 0.083333(18.850n-2.8584) \text{ and } n\in\mathbb{Z}$$

- 22 pairs from 38
  - 8 pairs – very unexpected
  - 14 pairs – quite unexpected

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Correctness of the students' answer /
the students' opinion about correctness of their answer

	Stud. correct	Stud. partially correct	Stud. incorrect	Stud. do not know	
Math. correct	6	4	2		12
Math. incorrect	6	1	5	4	16
Math. particular/general	2	3			5
Math. incorrect → correct	3		1		4
	17	8	8	4	

- Even after seeing the correct answer produced by CAS the 6 pairs marked their wrong answer as correct

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Correctness of the students' answer /
the students' opinion about correctness of their answer

	Stud. correct	Stud. partially correct	Stud. incorrect	Stud. do not know	
Math. correct	6	4	2		12
Math. incorrect	6	1	5	4	16
Math. particular/general	2	3			5
Math. incorrect → correct	3		1		4
	17	8	8	4	

- CAS answer was so confusing to some students that they marked their right answer as wrong (2 pairs)

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Correctness of the students' answer /
the students' opinion about correctness of their answer

	Stud. correct	Stud. partially correct	Stud. incorrect	Stud. do not know	
Math. correct	6	4	2		12
Math. incorrect	6	1	5	4	16
Math. particular/general	2	3			5
Math. incorrect → correct	3		1		4
	17	8	8	4	

- 4 pairs got the correct answer after changes with a green pen
  - *Rate the correctness of your (possibly corrected) answer*

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Correctness of the students' answer /
the students' opinion about correctness of CAS answer

	Stud. CAS correct	Stud. CAS partially correct	Stud. CAS incorrect	Stud. CAS do not know	
Math. correct	8			4	12
Math. incorrect	10			7	17
Math. particular/general	5				5
Math. incorrect → correct	4				4
	27	0	0	11	

- Nobody marked that the CAS answer is wrong

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Correctness of the students' answer /
the students' opinion about correctness of CAS answer

	Stud. CAS correct	Stud. CAS partially correct	Stud. CAS incorrect	Stud. CAS do not know	
Math. correct	8			4	12
Math. incorrect	10			7	17
Math. particular/general	5				5
Math. incorrect → correct	4				4
	27	0	0	11	

- 27 pairs chose that CAS answer is correct

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Correctness of the students' answer /
the students' opinion about correctness of CAS answer

	Stud. CAS correct	Stud. CAS partially correct	Stud. CAS incorrect	Stud. CAS do not know	
Math. correct	8			4	12
Math. incorrect	10			7	17
Math. particular/general	5				5
Math. incorrect → correct	4				4
	27	0	0	11	

- 11 pairs did not know

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Equivalence of the students' and CAS answer / the students' opinion about equivalence of their and CAS answer

	Stud. equivalent	Stud. non-equivalent	Stud. do not know	
Math. equivalent	5	4	3	12
Math. non- equivalent	4	9	4	17
Math. particular/general	2	2	1	5
Math. non-equivalent → equivalent	1	2	1	4
	12	17	9	

- Only 5 pairs found that their answer is equivalent with the CAS answer (when it was actually equivalent)

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Equivalence of the students' and CAS answer / the students' opinion about equivalence of their and CAS answer

	Stud. equivalent	Stud. non-equivalent	Stud. do not know	
Math. equivalent	**5**	**4**	**3**	**12**
Math. non- equivalent	**4**	**9**	**4**	**17**
Math. particular/general	**2**	**2**	**1**	**5**
Math. non-equivalent → equivalent	**1**	**2**	**1**	**4**
	12	**17**	**9**	

- 7 pairs did not notice the equivalence

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

Equivalence of the students' and CAS answer / the students' opinion about equivalence of their and CAS answer

	Stud. equivalent	Stud. non-equivalent	Stud. do not know	
Math. equivalent	**5**	**4**	**3**	**12**
Math. non- equivalent	**4**	**9**	**4**	**17**
Math. particular/general	**2**	**2**	**1**	**5**
Math. non-equivalent → equivalent	**1**	**2**	**1**	**4**
	12	**17**	**9**	

- 4 pairs decided that the answers are equivalent while the students' answer and the CAS answer were not equivalent

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

- What are the differences between your answer and the CAS answer?

- 27 pairs noticed the important differences between answers
  - radians/degrees
  - one/two answers
  - particular/general solution

Results - $\sin(4x+2)=\frac{\sqrt{3}}{2}$

- How are your answer and the CAS answer related (analyse equivalence/non-equivalence, particular solutions/general solutions)?

- As the answers are quite different looking it would be natural to investigate the CAS answer and try to derive one from another
- It seems that the students were not keen on deeper exploration
- The students did not provide any analysis on the relations between the answers
- An analysis of the relation between the students' answer and the CAS answer
  - 2 pairs – sufficient
  - 5 pairs – insufficient
  - 23 pairs – almost missing
  - 8 pairs – completely missing
- Deeper analyzes of the discussion tapes could proovide a better overview of the process of how they discuss on differences and relations

Results - $2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^{\circ};0^{\circ}]$

- The second equation was solved a little bit better than the first one
- 19 pairs solved it correctly

$$-\frac{\pi}{12} \qquad -15^{\circ}$$

- 15 pairs solved it incorrectly
  - unfinished solutions
  - different mistakes
- 3 pairs provided general solution while particular solution was asked

Results - $2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^\circ;0^\circ]$

- How unexpected is the CAS answer at first view?

school answer

$-\frac{\pi}{12} \qquad -15^\circ$

general school answer

$$x=\pm\frac{\pi}{4}+n\pi, n\in Z$$

$$x=(-1)^{n+1}\frac{\pi}{12}+\frac{n\pi}{2}, n\in Z$$

CAS answer

$$x=\pi\left(n-\frac{1}{4}\right)\approx 3.1416\,(n-0.25000) \text{ and } n\in\mathbb{Z}$$

$$x=\pi\left(n+\frac{1}{4}\right)\approx 3.1416\,(n+0.25000) \text{ and } n\in\mathbb{Z}$$

$$x=\pi\left(n-\frac{1}{12}\right)\approx 3.1416\,(n-0.083333) \text{ and } n\in\mathbb{Z}$$

$$x=\pi\left(n+\frac{7}{12}\right)\approx 3.1416\,(n+0.58333) \text{ and } n\in\mathbb{Z}$$

- 12 pairs from 38
  - 2 pairs – very unexpected
  - 10 pairs – quite unexpected
- 18 pairs from 38
  - 6 pairs – very expected
  - 12 pairs – quite expected

Results - $2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^\circ;0^\circ]$

Correctness of the students' answer /
the students' opinion about correctness of their answer

	Stud. correct	Stud. partially correct	Stud. incorrect	Stud. do not know	
Math. correct	17	1	1		19
Math. incorrect	2	9	2	1	14
Math. particular/general	1	1		1	3
Math. incorrect → correct	1				1
	21	11	3	2	

- Most of the pairs marked their answers as correct or partially correct. Even pairs with wrong answers chose the option "correct answer" and "partially correct"

Results - $2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^\circ;0^\circ]$

Correctness of the students' answer /
the students' opinion about correctness of CAS answer

	Stud. CAS correct	Stud. CAS partially correct	Stud. CAS incorrect	Stud. CAS do not know	
Math. correct	11	5		3	19
Math. incorrect	10	1	1	2	14
Math. particular/general	1	1		1	3
Math. incorrect → correct	1				1
	23	7	1	6	

- Only one pair thought the CAS answer to this equation was wrong. However, the answer of these students was incorrect

Results - $2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^\circ;0^\circ]$

Correctness of the students' answer /
the students' opinion about correctness of CAS answer

	Stud. CAS correct	Stud. CAS partially correct	Stud. CAS incorrect	Stud. CAS do not know	
Math. correct	11	5		3	19
Math. incorrect	10	1	1	2	14
Math. particular/general	1	1		1	3
Math. incorrect → correct	1				1
	23	7	1	6	

- Most of the pairs marked the CAS answer as correct or partially correct

Results - $2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^\circ;0^\circ]$

Correctness of the students' answer /
the students' opinion about correctness of CAS answer

	Stud. CAS correct	Stud. CAS partially correct	Stud. CAS incorrect	Stud. CAS do not know	
Math. correct	11	5		3	19
Math. incorrect	10	1	1	2	14
Math. particular/general	1	1		1	3
Math. incorrect → correct	1				1
	23	7	1	6	

- Some students marked that it is hard to say is this answer wrong or correct or partially correct as it is not possible to request particular solution

Results - $2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^{\circ};0^{\circ}]$

Equivalence of the students' and CAS answer / the students' opinion about equivalence of their and CAS answer

	Stud. equivalent	Stud. non-equivalent	Stud. do not know	
Math. equivalent	1		2	3
Math. non- equivalent	4	4	6	14
Math. particular/general	10	6	3	19
Math. non-equivalent → particular/general	1			
	16	10	11	

- Most of the students chose the equivalence. And this is not wrong choice because general solutions were equivalent in 12 cases

Results - $2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^{\circ};0^{\circ}]$

Equivalence of the students' and CAS answer / the students' opinion about equivalence of their and CAS answer

	Stud. equivalent	Stud. non-equivalent	Stud. do not know	
Math. equivalent	1		2	3
Math. non- equivalent	4	4	6	14
Math. particular/general	10	6	3	19
Math. non-equivalent → particular/general	1			
	16	10	11	

- 10 pairs marked the non-equivalence. And again this is not wrong choice as particular answer and general answer are not equivalent.

Results - $2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^{\circ};0^{\circ}]$

- What are the differences between your answer and the CAS answer?
- How are your answer and the CAS answer related (analyse equivalence/non-equivalence, particular solutions/general solutions)?

- Only 2 pairs checked whether their answer really matched the CAS answer
- 18 pairs indicated that their solution is in concrete interval but the CAS solution is not
- 3 pairs mentioned that CAS has more solutions
- 1 pair noticed the relation between the general and particular solution
- Other 14 pairs did not write any reasonable description

Results - $\frac{\tan^2 x}{\tan x} = 0$

- 19 pairs solved it correctly

  no solutions

- 18 pairs solved it incorrectly

  $\tan x = 0$

  $x = n\pi, n \in Z$

Results - $\frac{\tan^2 x}{\tan x} = 0$

- How unexpected is the CAS answer at first view?

school answer

no solutions

CAS answer

$x = \pi n \approx 3.1416\, n$ and $n \in \mathbb{Z}$

- 18 pairs - unexpected
  - 15 pairs – the correct answer
- 15 pairs - expected
  - 13 pairs – the wrong answer

Results - $\frac{\tan^2 x}{\tan x} = 0$

Correctness of the students' answer /
the students' opinion about correctness of their answer

	Stud. correct	Stud. partially correct	Stud. incorrect	Stud. do not know	
Math. correct	15	1		1	17
Math. incorrect	12	3		1	16
Math. correct → incorrect		1	1		2
Math. incorrect → CAS like incorrect	1				1
	28	5	1	2	

- The larger majority of both groups (with correct answer as well with wrong answer) had marked their answer as correct in worksheet after seeing the CAS answer

Results - $\frac{\tan^2 x}{\tan x} = 0$

Correctness of the students' answer /
the students' opinion about correctness of their answer

	Stud. correct	Stud. partially correct	Stud. incorrect	Stud. do not know	
Math. correct	15	1		1	17
Math. incorrect	12	3		1	16
Math. correct → incorrect		1	1		2
Math. incorrect → CAS like incorrect	1				1
	28	5	1	2	

- 2 pairs changed their correct answer to incorrect answer
- One pair had a completely wrong answer, which was not equivalent to CAS answer. After seeing the CAS answer they corrected their incorrect answer to another incorrect answer, but this one was equivalent to the CAS answer

Results - $\frac{\tan^2 x}{\tan x} = 0$

Correctness of the students' answer /
the students' opinion about correctness of CAS answer

	Stud. CAS correct	Stud. CAS partially correct	Stud. CAS incorrect	Stud. CAS do not know	
Math. correct	3		12	4	19
Math. incorrect	13	1			14
Math. correct → incorrect	1			1	
Math. incorrect → CAS like incorrect	1				
	18	1	12	5	

- Only 12 pairs marked that the CAS answer is wrong. All these pairs had the correct solution

Results - $\frac{\tan^2 x}{\tan x} = 0$

Correctness of the students' answer /
the students' opinion about correctness of CAS answer

	Stud. CAS correct	Stud. CAS partially correct	Stud. CAS incorrect	Stud. CAS do not know	
Math. correct	3		12	4	19
Math. incorrect	13	1			14
Math. correct → incorrect	1			1	
Math. incorrect → CAS like incorrect	1				
	18	1	12	5	

- 3 pairs with the right answer had marked that the CAS answer is correct

Results - $\frac{\tan^2 x}{\tan x} = 0$

Correctness of the students' answer /
the students' opinion about correctness of CAS answer

	Stud. CAS correct	Stud. CAS partially correct	Stud. CAS incorrect	Stud. CAS do not know	
Math. correct	3		12	4	19
Math. incorrect	13	1			14
Math. correct → incorrect	1			1	
Math. incorrect → CAS like incorrect	1				
	18	1	12	5	

- Of course, those students, who had the incorrect solution (same as CAS), thought that the CAS answer is correct

Results - $\frac{\tan^2 x}{\tan x} = 0$

Equivalence of the students' and CAS answer / the students' opinion about equivalence of their and CAS answer

	Stud. equivalent	Stud. non-equivalent	Stud. do not know	
Math. equivalent	**13**			**13**
Math. non- equivalent	**1**	**17**	**1**	**19**
Math. particular/general		**1**		**1**
Math. non-equivalent → equivalent	**2**	**1**		**3**
	16	**19**	**1**	

- In case of this equation the students identified the equivalence and non-equivalence of their answer and the CAS answer very well
  - If the answers were equivalent, the students were able to detect this
  - If they were not equivalent then students selected non-equivalence as well

Results - Over the three equations

Adequate identification of correctness of students' answer

Equation	Adequate identification
$\sin(4x+2)=\frac{\sqrt{3}}{2}$	**45%**
$2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^\circ;0^\circ]$	**57%**
$\frac{\tan^2 x}{\tan x}=0$	**44%**

- We count the opinion adequate if
  - Stud. correct and Math. correct
  - Stud. incorrect and Math. incorrect
  - Stud. partially correct and Math. particular/general
  - Stud. correct and Math. incorrect -> correct

Results - Over the three equations

Adequate identification of correctness of CAS answer

Equation	Adequate identification
$\sin(4x+2)=\frac{\sqrt{3}}{2}$	**71%**
$2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^\circ;0^\circ]$	**19%**
$\frac{\tan^2 x}{\tan x}=0$	**33%**

$\sin(4x+2)=\frac{\sqrt{3}}{2}$ correct

$2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^\circ;0^\circ]$ partially correct

$\frac{\tan^2 x}{\tan x}=0$ incorrect

Results - Over the three equations

Adequate identification of equivalence/non-equivalence

Equation	Adequate identification
$\sin(4x+2)=\frac{\sqrt{3}}{2}$	**39%**
$2\sin 2x\cos 2x+\cos 2x=0 \quad [-30^\circ;0^\circ]$	**13%**
$\frac{\tan^2 x}{\tan x}=0$	**89%**

- We count the opinion adequate if
  - Stud. equivalent and Math. equivalent
  - Stud. non-equivalent and Math. non-equivalent
  - Stud. equivalent and Math. non-equivalent -> equivalent

Conclusion

What will happen when students themselves are encouraged to analyse differences, equivalence and correctness of their own answers and CAS answers?

- A very easy answer is that they can work on trigonometry for a whole lesson
- The task seemed to be new for the students
  - Usually, only the solution of an equation is needed and not more
  - The format seemed to be interesting and catching
- These somewhat unexpected answers could support the discussion and provide a possibility to activate students
- In fact, the role of the teacher was mainly to introduce the lesson and to answer some questions during the lesson

Conclusion

What differences do the students notice foremost?

- The outstanding differences, like radians/degrees, particular/general solution, were noticed quite often
- Sometimes the number of series is quite confusing as the answers based on different formulae

Conclusion

How do the students understand correctness of the answers?

- It was somewhat complicated, as they have not been given the school correct answer
- Many students seemed to trust CAS while others trusted themselves
- It seems that it depends on the students' confidence in their answer whether they trust the system or themselves
- Furthermore, correctness could sometimes be complicated to evaluate, for example in case of particular and general solution

Conclusion

Are students able to ascertain equivalence/non-equivalence?

- It depends very much on the equation and the task
- If the students' answer and the CAS answer looked quite different, then students did not know how to determine equivalence/non-equivalence and quite often did it incorrectly
- Equivalence/non-equivalence of particular and general solutions is very questionable
- However in case of some equations, where the CAS answer is the same as a possible student answer or very different (like some solution /no solutions), it can be done very well

Conclusion

How do the students explain equivalence/non-equivalence?

- If the students' answer and the CAS answer looked quite different, then the students usually did not explain equivalence/non-equivalence at all
  - It is maybe one of the important message for improving the worksheet. A thorough comparison of different answers would be very instructive, for example, for understanding general solutions properly, including role of the n
  - Probably, more detailed subtasks would be useful
- It could be also a good idea to use equations where the answer is more similar to the school (and hopefully the students') answer

Conclusion

Are there any differences in this regard between different types of equations and answers?

- The choice of equation (particularly by answer) is very important
- It would be useful to evaluate the "distance" between the CAS answer and school answer (or probable students' answer)
  - For example, the "distance" seems to be too large in case of the first equation

Further work

- This paper is based mainly on the worksheets
- It seems that the students did not make enough use of the provided opportunity to analyze differences, equivalence and correctness of their own answers and the CAS answers in writing
- It is possible that students simply discussed these questions and answers and did not mark their thoughts on paper
- This could be clarified using the audio-tapes and certainly this would be the next step in our study