Advanced techniques to compute improper integrals using a CAS

G. Aguilera\textsuperscript{1}, J.L. Galán\textsuperscript{1}, M.Á. Galán\textsuperscript{1}, Y. Padilla\textsuperscript{1}, P. Rodríguez\textsuperscript{1}, R. Rodríguez\textsuperscript{2}

\textsuperscript{1}University of Málaga, Spain
\textsuperscript{2}University of Madrid, Spain

Technology and its integration in Mathematics Education

\textbf{TiME} 2014
July 1-5. Krems, Austria
1 Introduction
   - Motivation of the problem
   - Our approach
   - Rule-Based Integrators

2 Theoretical frames
   - Laplace Transform $\mathcal{L}$
   - Fourier Transform $\mathcal{F}$
   - The Residue Theorem

3 Examples
   - Laplace Transform
   - Fourier Transform
   - The Residue Theorem
Motivation of the problem

Let us consider the following types of improper integrals:

\[
\int_{0}^{\infty} f(t) \, dt \quad ; \quad \int_{-\infty}^{0} f(t) \, dt \quad \text{and} \quad \int_{-\infty}^{\infty} f(t) \, dt
\]

Let \( F \) be an antiderivative of \( f \). The basic approach to compute such integrals involves the following computations:
Motivation of the problem

\[ \int_{0}^{\infty} f(t) \, dt = \lim_{m \to \infty} \int_{0}^{m} f(t) \, dt = \lim_{m \to \infty} \left( F(m) - F(0) \right) \]

\[ \int_{-\infty}^{0} f(t) \, dt = \lim_{m \to -\infty} \int_{m}^{0} f(t) \, dt = \lim_{m \to -\infty} \left( F(0) - F(m) \right) \]

\[ \int_{-\infty}^{\infty} f(t) \, dt = \int_{-\infty}^{0} f(t) \, dt + \int_{0}^{\infty} f(t) \, dt \quad \text{or, if convergence,} \]

\[ \int_{-\infty}^{\infty} f(t) \, dt = \lim_{m \to \infty} \int_{-m}^{m} f(t) \, dt = \lim_{m \to \infty} \left( F(m) - F(-m) \right) \]

(Cauchy principal value)
Motivation of the problem

But, what happens if an antiderivative $F$ for $f$ or the above limits do not exist?

For example, for

$$\int_0^\infty \frac{\sin(at)}{t} \, dt \quad ; \quad \int_0^\infty \frac{\cos(at) - \cos(bt)}{t} \, dt \quad \text{or} \quad \int_{-\infty}^\infty \frac{\cos(bt)}{t^2 + a^2} \, dt$$

the antiderivatives can not be computed. Hence, the above procedures cannot be used for these examples.
In this work we will deal with advance techniques to compute this kind of improper integrals using a CAS.

Laplace and Fourier transforms or the Residue Theorem in Complex Analysis are some advance techniques which can be used for this matter.

We will introduce the file ImproperIntegrals.mth, developed in Derive 6, which deals with such computations.
Some CAS use different rules for computing integrations.

For example Rubi system, a rule-based integrator developed by Albert Rich (see http://www.apmaths.uwo.ca/~arich/), is a very powerful system for computing integrals using rules.

We will be able to develop new rules schemes for some improper integrals using ImproperIntegrals.mth.

These new rules can extend the types of improper integrals that a CAS can compute.
To achieve this goal, we first establish the theoretical frames. Specifically, we will use:

- Laplace Transforms
- Fourier Transforms
- The Residue Theorem
We define the **Laplace Transform** of a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$\mathcal{L} \left[ f(t) \right] = F_\mathcal{L}(s) = \int_0^\infty e^{-st} f(t) \, dt$$

if such integral exists.
Improper integrals computation using Laplace Transform

According with the previous definition, evaluating in \( s = 0 \):

\[
F_L(0) = \int_0^\infty f(t) \, dt
\]

But normally, Laplace Transform exists for \( s > 0 \). Therefore, in order to compute an improper integral we can use the following result:

\[
\int_0^\infty f(t) \, dt = \lim_{s \to 0^+} \mathcal{L}[f(t)]
\]

if both, the Laplace Transform and the limit exist.
We define the **Fourier Transform** of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ as

$$
\mathcal{F}[f(t)] = F_{\mathcal{F}}(s) = \int_{-\infty}^{\infty} f(t) e^{-ist} \, dt \quad s \in \mathbb{R}
$$

if such integral exists.
According with the previous definition, evaluating in $s = 0$:

$$\int_{-\infty}^{\infty} f(t) \, dt = \mathcal{F}[f(t)] \bigg|_{s=0}$$

if the Fourier Transform exists.
The Residue Theorem

Let $C$ be a closed piecewise smooth positive oriented curve.

Let $f$ be an analytic function with a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ inside $C$.

Let $\text{Res}_{z=z_k} f(z)$ be the residue of $f$ in $z_k$.

Then:

$$\oint_C f(z) \, dz = 2\pi i \left( \text{Res}_{z=z_1} f(z) + \text{Res}_{z=z_2} f(z) + \cdots + \text{Res}_{z=z_n} f(z) \right)$$

$$= 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$$
Let $f$ be an analytic function with a finite set of isolated singularities $S_P$ inside $\mathcal{P} \equiv \Im(z) \geq 0$ none of them being on the real axis.

Let $\text{CPV}(I)$ the Cauchy Principal Value of integral $I$

Then:

1. If $\lim_{z \to \infty} z f(z) = 0$ \implies

$$\text{CPV} \left( \int_{-\infty}^{\infty} f(x) \, dx \right) = \oint_{\mathcal{P}} f(z) \, dz = 2\pi i \sum_{z_k \in S_P} \text{Res}_{z=z_k} f(z).$$
Improper integrals computation using the Residue Theorem

2. If \( \lim_{z \to \infty} f(z) = 0 \implies \)

\[
\text{CPV} \left( \int_{-\infty}^{\infty} f(x) \cos(ax) \, dx \right) = \text{Re} \left( \oint_{\mathcal{P}} f(z) e^{iaz} \, dz \right) \\
= \text{Re} \left( 2\pi i \sum_{z_k \in S_\mathcal{P}} \text{Res}_{z=z_k} f(z) e^{iaz} \right)
\]

\[
\text{CPV} \left( \int_{-\infty}^{\infty} f(x) \sin(ax) \, dx \right) = \text{Im} \left( \oint_{\mathcal{P}} f(z) e^{iaz} \, dz \right) \\
= \text{Im} \left( 2\pi i \sum_{z_k \in S_\mathcal{P}} \text{Res}_{z=z_k} f(z) e^{iaz} \right)
\]
Improper integrals computation using Laplace Transform

- \[ \mathcal{L} \left[ \frac{\sin(at)}{t} \right] = \frac{\pi}{2} - \tan^{-1} \left( \frac{s}{a} \right) \quad \Rightarrow \]

\[ \int_0^\infty \frac{\sin(at)}{t} \, dt = \lim_{s \to 0^+} \left( \frac{\pi}{2} - \tan^{-1} \left( \frac{s}{a} \right) \right) = \frac{\pi}{2} \]

- \[ \mathcal{L} \left[ \frac{\cos(at) - \cos(bt)}{t} \right] = \ln \left( \sqrt{\frac{s^2 + b^2}{s^2 + a^2}} \right) \quad \Rightarrow \]

\[ \int_0^\infty \frac{\cos(at) - \cos(bt)}{t} \, dt = \lim_{s \to 0^+} \ln \left( \sqrt{\frac{s^2 + b^2}{s^2 + a^2}} \right) = \ln \left( \left| \frac{b}{a} \right| \right) \]
Improper integrals computation using Fourier Transform

- \( \mathcal{F} \left[ e^{-|a|t^2} \right] = \sqrt{\frac{\pi}{|a|}} e^{-\frac{s^2}{4|a|}} \implies \)
  \[
  \int_{-\infty}^{\infty} e^{-|a|t^2} \, dt = \left. \sqrt{\frac{\pi}{|a|}} e^{-\frac{s^2}{4|a|}} \right|_{s=0} = \sqrt{\frac{\pi}{|a|}}
  \]

- \( \mathcal{F} \left[ \frac{\cos(bt)}{a^2 + t^2} \right] = \frac{\pi}{2|a|} \left( e^{-|a| \cdot |s-b|} + e^{-|a| \cdot |s+b|} \right) \implies \)
  \[
  \int_{-\infty}^{\infty} \frac{\cos(bt)}{a^2 + t^2} \, dt = \left. \frac{\pi}{2|a|} \left( e^{-|a| \cdot |s-b|} + e^{-|a| \cdot |s+b|} \right) \right|_{s=0} = \frac{\pi \cdot e^{-|ab|}}{|a|}
  \]
Improper integrals computation using the Residue Theorem
Advanced techniques to compute improper integrals using a CAS

G. Aguilera¹, J.L. Galán¹, M.Á. Galán¹,
Y. Padilla¹, P. Rodríguez¹, R. Rodríguez²

¹University of Málaga, Spain
²University of Madrid, Spain

Technology and its integration in Mathematics Education

TiME 2014
July 1-5. Krems, Austria