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Modeling a Real Pendulum with Smartphone Sensor Technology

TSG 21: Mathematical applications and modelling in the teaching and learning of mathematics

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Abstract

Smartphones are often considered to pose a threat to education, yet some of their features provide valuable benefits to solving mathematical modelling problems. New smartphone sensor technology allows to test mathematical models of physical systems in the classroom. This feedback from real measurements can be used to improve and refine the mathematical model, thereby completing the modelling cycle. We present an example where we use a smartphone as a pendulum bob and indirectly measure the elongation angle with the smartphone's built-in acceleration sensors. A spreadsheet implementation of the classical Runge-Kutta algorithm is then used to compute approximate solutions of the differential equation representing the swinging pendulum. Numerical analysis of differential equations is a characteristic and essential element of applied mathematics in science and industry that should get more attention in school mathematics. Assisted by technology, solving real-world problems is absolutely achievable and could be very motivating for students.

Keywords:	*Schlüsselwörter:*
Numerical methods	Numerische Methoden
Applications in physics	Physikalische Anwendungen
Smartphone sensors	Smartphone-Sensoren

1 Introduction

In many Western countries, the public image of mathematics is poor. In general, higher mathematical education is not recognized as being very desirable nor as being of particular value or importance. Even worse, mathematics is the only subject in the school curriculum that adults are "proud" to have been weak in during their school years (Posamentier, 2015). Part of an explanation could be related to the fact that most people simply do not have any idea of what professional mathematicians do and in how many branches of industry highest-level mathematics plays a role. Achievements and typical problems of industrial or professional mathematics are not only practically invisible in the media, they are also hardly ever mentioned in school. Although the mathematical details of real-world applications would most of the time even exceed the teacher's knowledge, there will always be certain aspects of such problems that are comprehensible for students as well. In fact, the basic principles of some characteristic features of industrial mathematics are not so far away from school mathematics. Presenting them in school could provide students a glimpse on what professional mathematicians really do and change their opinion about mathematics for the better (Maaß, 2015). Numerical analysis of differential equations is a very typical element of applied mathematics that could easily be included in mathematics education.

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2 Differential equations in school - why not?

Differential equations are of utmost importance in applied mathematics. The fundamental theories of physics are formulated as (systems of partial) differential equations, among them the Maxwell equations describing the propagation of electromagnetic waves, the Schrödinger equation in quantum mechanics, or the Einstein equations in general relativity. In fact, differential equations play an important role not only in physics, but in all natural sciences as well as in engineering, finance, economy, and even medicine. In contrast to their ubiquity in problems of applied mathematics, differential equations are not very strongly represented in school curriculums. This might be related to the fact that there are just a few examples of differential equations that can be solved within the framework of school mathematics. In school, the equation of exponential growth or decay $\dot{x}(t) = \beta \cdot x(t)$ is probably the standard example that is considered (here $\dot{x}(t) = dx/dt$). Together with the equation of a freely falling body in vacuum $\ddot{x}(t) = g$, it is often the only differential equation that is taught (if differential equations are mentioned at all). The harmonic oscillator equation $\ddot{x}(t) = -k \cdot x(t)$ represents another example that might occasionally be discussed in school, but all more complicated differential equations are usually ignored, since the analytical methods to solve them would go far beyond school mathematics. However, there is no need to actually solve a differential equation in order to have fun with it.

3 Numerical solution methods for differential equations

Despite their great importance in industrial mathematics, numerical approximation methods for differential equations are often underrepresented and marginalized in school mathematics. By using numerical methods we can always construct approximate solutions, even for the "most ugly" differential equations. While this was extremely tedious work in the pre-computer era, numerical algorithms can now easily be implemented in any spreadsheet software. Moreover, authentic modeling problems in applied mathematics very often involve differential equations that can only be tackled by numerical methods anyway.

For convenience of the reader we want to briefly review the classical Runge-Kutta method (RK4), a numerical algorithm for ordinary differential equations. We consider the initial value problem

$$\dot{x}(t) = F(t, x(t)) \qquad x(t_0) = x_0 \,,$$

for which we want to iteratively compute an approximation of the solution. Picking a time step of size $s > 0$ such that $t_{n+1} = t_n + s$, we define

$$x_{n+1} = x_n + \tfrac{s}{6}\left(k_{1,n} + 2k_{2,n} + 2k_{3,n} + k_{4,n}\right) ,$$

where $x_n := x(t_n)$ and

$$k_{1,n} = F(t_n, x_n),$$

$$k_{2,n} = F\left(t_n + \frac{s}{2}, x_n + \frac{s}{2}k_{1,n}\right),$$

$$k_{3,n} = F\left(t_n + \frac{s}{2}, x_n + \frac{s}{2}k_{2,n}\right),$$

$$k_{4,n} = F\left(t_n + s, x_n + sk_{3,n}\right).$$

This numerical scheme can be programmed in every standard spreadsheet software.

4 Modeling a real pendulum using smartphone technology

Over the last years, applications and modeling have become more and more important in mathematics education. Yet, there are seldom opportunities to test (and subsequently refine) a mathematical model by performing experiments on the real-world system the model is meant to describe. However, it is often overlooked that smartphones actually comprise high-precision measuring instruments that can be used in many ways to collect real data for physical systems. Rapid advances in technology made it possible that today practically all smartphones are equipped with acceleration sensors, magnetic field sensors, sensors for luminosity and air pressure, as well as GPS receivers. All these sensors can be used to make physical measurements and get real data.

Using a smartphone as a pendulum bob (for example by putting it in a box and fixing this box to a string, see Figure 1), the acceleration sensors measure the acceleration of the bob while it is swinging. There are various

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free sensor apps available to record the data and save them in CSV format. The data file can then be transferred to a PC and imported into a spreadsheet. Setting up this experiment does not require much preparation and performing it in class may constitute an exciting hands-on experience for the students. The points in the plot shown in Figure 2 represent the acceleration data acquired by letting a smartphone pendulum bob swing on a string of 2.5 meters length. Of course, the measured data can be analyzed mathematically and statistically on their own right (e.g. data interpolation, determine the oscillation period, finding a function that best represents the data or the envelope, etc.). This task alone provides a nice exercise, especially with the aid of technology (any spreadsheet software will do).

However, modeling the pendulum as a differential equation and computing approximate solutions that can then be compared with the measurements is even more interesting and promising. In addition, this problem opens up a wonderful playground for interdisciplinary teaching, since it requires substantial input from physics and some basic programming skills. There are various stages of detail and accuracy for modeling a pendulum, some of which we want to briefly review.

The differential equation of an idealized pendulum can be obtained by considering a point mass *m* swinging on a massless rod of length *L*.

Figure 1: Left and right: Schematic drawings of a pendulum showing the relevant acceleration components. Middle: Smartphone in a plastic box, used as the bob of a pendulum.

Denoting the tangential acceleration by a_T, we have $m \cdot a_T = -m \cdot g \cdot \sin\alpha$ by Newton's second law (neglecting any effects of friction or air resistance). Since the tangential acceleration is related to the elongation angle by $a_T = L \cdot \ddot{\alpha}$, we obtain $L \cdot \ddot{\alpha}(t) + g \cdot \sin\alpha\,(t) = 0$. This differential equation cannot be solved in terms of elementary functions, but for small angles we may use the approximation $\sin\alpha \approx \alpha$ (in radiant measure), thereby arriving at $\ddot{\alpha}(t) + \omega^2 \cdot \alpha(t) = 0$ (where $\omega^2 = g/L$). This is the equation of the harmonic oscillator, whose solutions are harmonic oscillations $\alpha(t) = A \cdot \cos(\omega t + \varphi)$. Here A denotes the amplitude and ω is the angular frequency.

As already indicated, this model is a crude simplification of a real pendulum. In reality, the pendulum will slow down as a result of air resistance and friction in the bearing. Moreover, for larger elongation angles, the approximation $\sin\alpha \approx \alpha$ is no longer justified and the form of the oscillations will significantly deviate from a harmonic behavior. A realistic model for a pendulum, in particular including air resistance, would be

$$\ddot{\alpha}(t) + c_1 \cdot \sin\alpha\,(t) + c_2 \cdot \alpha(t) \cdot |\alpha(t)| = 0$$

where c_1 and c_2 are constants determined by the geometry and the masses of the bob and the rod. This differential equation is much more complicated than the one for the harmonic oscillator. In fact, it is virtually unsolvable by analytical methods. However, if we only want to find an approximate solution, it does not make any difference, whether we work with the idealized and simple harmonic oscillator equation or the realistic but much more complicated model equation. We just have to modify the right-hand side F in our Runge-Kutta algorithm accordingly.

5 Conclusion

We believe that smartphone sensors can be used in various ways to enrich mathematical modeling in school, in particular in the context of differential equations describing mechanical, optical, or electrical systems. With the help of a computer, it is not difficult to construct approximate solutions of differential equations, thereby conveying a more authentic picture of applied mathematics.

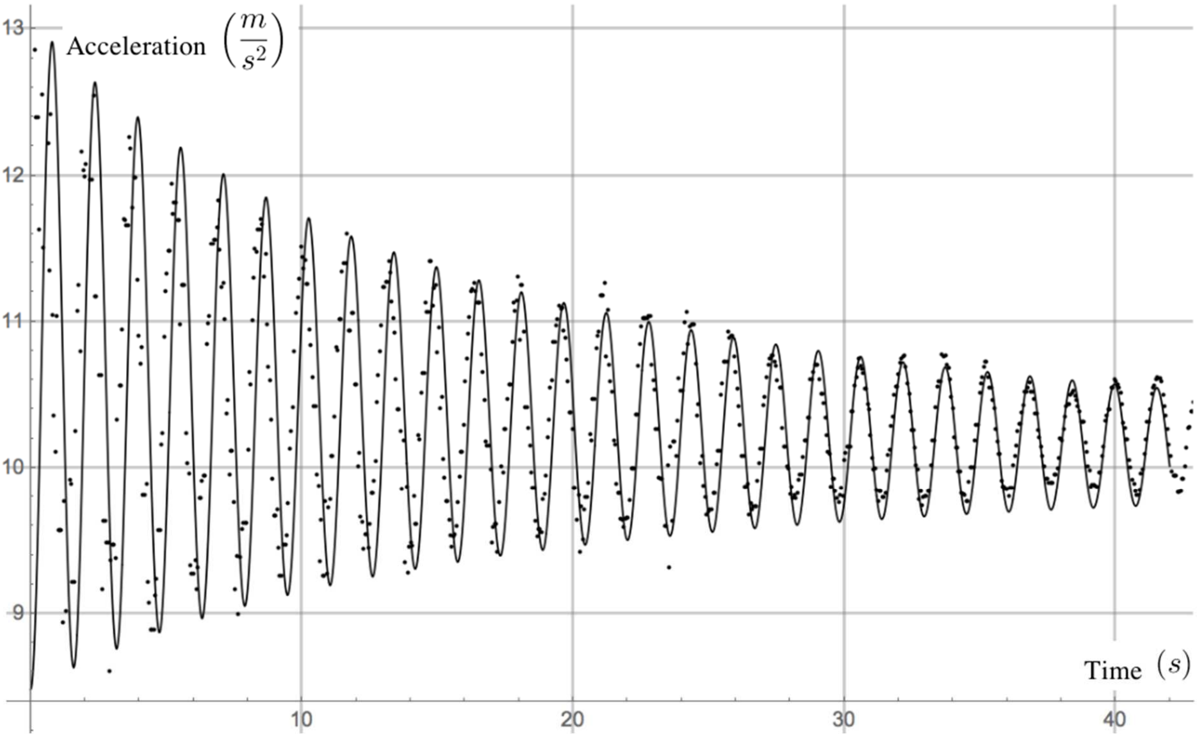

Figure 2: Plot of the measured smartphone acceleration (orthogonal to the smartphone screen) and the corresponding approximate solution of the differential equation that models the pendulum.

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