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PÄDAGOGISCHE HOCHSCHULE für NIEDERÖSTERREICH

Mathematics in Art, Architecture and Nature

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Abstract

Schools typically treat mathematics as an isolated subject rather than to integrate it with other areas of the school curriculum. This leaves mathematics as the subject, which accompanies students throughout all of their schooling, without any partner subjects. Here we will show how it manifests itself with nature, art and architecture. The article here is merely an introduction and provided with a hope that it will motivate awareness of mathematics beyond the school applications.

Keywords:	*Schlüsselwörter:*
Mathematics	Mathematik
Art	Kunst
Architecture	Architektur
Nature	Natur

1 Fibonacci Numbers

We begin in the year 1202 with the publication of the famous mathematics book Liber Abaci by the Italian mathematician Leonardo of Pisa (1175 – 1250), known today as Fibonacci. This book is largely famous for a problem posed in chapter 12 about the regeneration of rabbits.

However, before considering this problem, it should be noted that this book was the first publication that used the Hindu-Arabic numerals that we use today. The first sentence of the introduction is: “The nine Indian figures are: 9 8 7 6 5 4 3 2 1. With these nine figures, and with the sign 0, which the Arabs call zephyr, any number whatsoever is written, as demonstrated …” (Fibonacci used the term “Indian figures” for the Hindu numerals.) It took about another 50 years following the publication of this book that actual use of these new symbols was seen throughout parts of Europe.

Getting back to the problem of the regeneration of rabbits, we are asked to determine the number of rabbits existing each month at the end of one year following a specially define method of birthing. The list of the numbers of pairs of rabbits existing each month according to the procedure described for birth, generates the famous Fibonacci numbers. These numbers are: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,…., and the list taken from the book is shown in figure 1.

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Fig. 1: A copy of the original pages with the Fibonacci numbers.

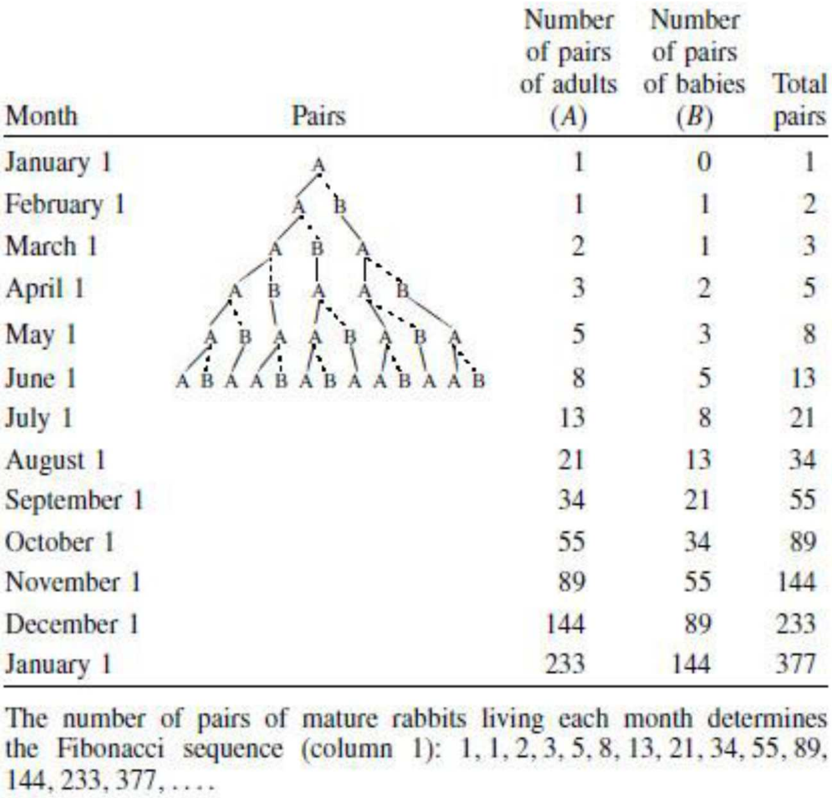
The statement of the problem is shown in figure 2.

Beginning 1 First 2 Second 3 Third 5 Fourth 8 Fifth 13 Sixth 21 Seventh 34	"A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also. Because the above written pair in the first month bore, you will double it; there will be two pairs in one month. One of these, namely the first, bears in the second month, and thus there are in the second month 3 pairs; of these in one month two are pregnant and in the third month 2 pairs of rabbits are born and thus there are 5 pairs in the month; in this month 3 pairs are pregnant and in the fourth month there are 8 pairs, of which 5 pairs bear another 5 pairs; these are added to the 8 pairs making 13 pairs in the fifth month; these 5 pairs that are born in this month do not mate in this month, but another 8 pairs are pregnant, and thus there are in the sixth month 21 pairs; to these are added the 13 pairs that are born in the seventh month; there will be 34 pairs in this month; to this are added the 21

Eighth 55 Ninth 89 Tenth 144 Eleventh 233 Twelfth 377	pairs that are born in the eighth month; there will be 55 pairs in this month; to these are added the 34 pairs that are born in the ninth month; there will be 89 pairs in this month; to these are added again the 55 pairs that are both in the tenth month; there will be 144 pairs in this month; to these are added again the 89 pairs that are born in the eleventh month; there will be 233 pairs in this month. To these are still added the 144 pairs that are born in the last month; there will be 377 pairs and this many pairs are produced from the above-written pair in the mentioned place at the end of one year. You can indeed see in the margin how we operated, namely that we added the first number to the second, namely the 1 to the 2, and the second to the third and the third to the fourth and the fourth to the fifth, and thus one after another until we added the tenth to the eleventh, namely the 144 to the 233, and we had the above-written sum of rabbits, namely 377 and thus you can in order find it for an unending number of months.”

Fig. 2: Translation from Fibonacci’s Liber Abaci.

To see how these rabbits are regenerated, we refer you to figure below, which shows how the pairs of rabbits increase and how the Fibonacci numbers were generated.



Month	Pairs	Number of pairs of adults (A)	Number of pairs of babies (B)	Total pairs
January 1	A	1	0	1
February 1	A B	1	1	2
March 1	A B A	2	1	3
April 1	A B A A B	3	2	5
May 1	A B A A B A B A	5	3	8
June 1	A B A A B A B A A B A A B	8	5	13
July 1		13	8	21
August 1		21	13	34
September 1		34	21	55
October 1		55	34	89
November 1		89	55	144
December 1		144	89	233
January 1		233	144	377

The number of pairs of mature rabbits living each month determines the Fibonacci sequence (column 1): 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

Fig. 3: Regeneration of rabbits from Fibonacci’s problem in Chapter 12 of Liber Abaci.

The next figure shows how the Fibonacci numbers can be generated independently of the famous rabbit problem. We begin with the numbers 1 and 1, and then add each the last pair to get the next number.

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$$
\begin{aligned}
&1\\
&1\\
&1+1=\mathbf{2}\\
&1+2=\mathbf{3}\\
&2+3=\mathbf{5}\\
&3+5=\mathbf{8}\\
&5+8=\mathbf{13}\\
&8+13=\mathbf{21}\\
&13+21=\mathbf{34}\\
&21+34=\mathbf{55}\\
&34+55=\mathbf{89}\\
&55+89=\mathbf{144}\\
&89+144=\mathbf{233}\\
&144+233=\mathbf{377}\\
&233+377=\mathbf{610}\\
&377+610=\mathbf{987}\\
&610+987=\mathbf{1597}\ \ldots
\end{aligned}
$$

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, …

Fig. 4: Recursive definition of the Fibonacci numbers

There are innumerable relationships that the Fibonacci numbers generate in mathematics. There is probably no field of mathematics, where the Fibonacci numbers cannot be shown to be included. One can have a lot of fun searching for patterns amongst the Fibonacci numbers or the relationship to other aspects of mathematics. It is interesting to note that since 1963, a journal issued four times a year and known as The Fibonacci Quarterly has presented a very wide selection of applications of these famous numbers. There are also many books written about the Fibonacci numbers, one of which I co-authored with Dr. Ingmar Lehmann, a professor at the Humboldt University in Berlin (Posamentier & Lehmann, 2007).

The Fibonacci numbers are also found in many aspects of nature. For example, if you count the spirals on a pine cone, you will find that in one direction, there are 8 spirals and any other direction, there are 13 spirals. Of course, 8 and 13 are Fibonacci numbers.

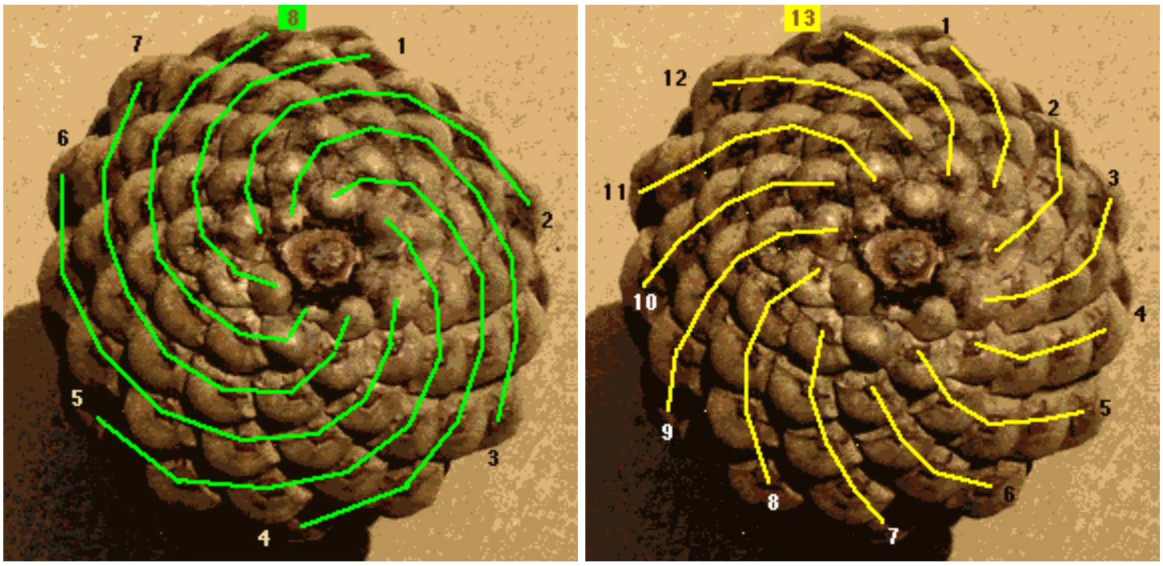

Fig. 5: Spiral arrangement of the bracts of a pine cone

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A pineapple also demonstrates the Fibonacci numbers, in that there are tracts determining spirals in 3 different directions: in one set of spirals we have 5 such spirals, in another one 8 spirals and a third one 13 spirals.

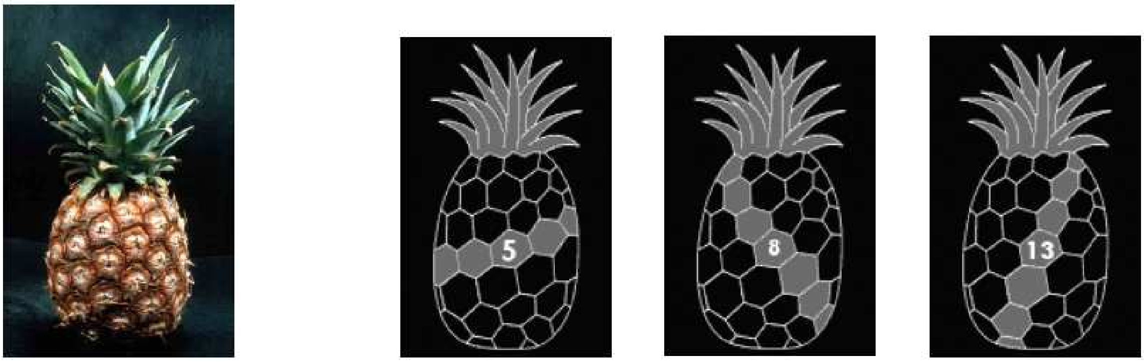

Fig. 6: Fibonacci numbers on a pineapple.

As we mentioned before, spirals can be found in many different plants, and you will be interested to notice that each time you count them, you will find the number of spirals in one of the directions is always a Fibonacci number. But nature produces more than just spirals. So, for example, we can create a spiral by tracking the branches up a tree. Beginning with the first branch and turning around the tree, each time touching only one branch higher to get to a branch, which is the same direction as the first branch you encountered, you will find that you will have passed a Fibonacci number of branches along the way.

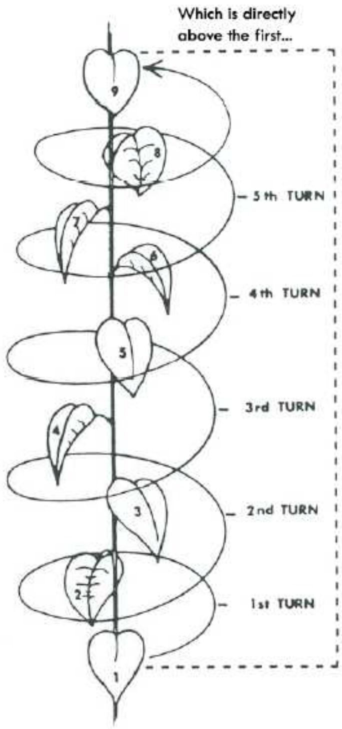

Fig. 7: The Phyllotaxis, or leaf arrangement.

2 The Golden Ratio

Moving along now from the Fibonacci numbers to one of the most important ratios in all of mathematics, known as “The Golden Ratio.” In the left column of the chart below, we show the ratio of consecutive Fibonacci numbers are generating a number approaching 1.61803… . And in the right column, the ratio approaches the number 0.61803…. For one thing, you will notice that these two ratios differ by 1. This is the only time that

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such a relationship exists, namely $x - \frac{1}{x} = 1$, which then leads to the equation: $x^2 - 1 = x$, or $x^2 - x - 1 = 0$, where *x* is the Golden Ratio.

$\frac{F_{n+1}}{F_n}$	$\frac{F_n}{F_{n+1}}$
$\frac{1}{1} = 1.000000000$	$\frac{1}{1} = 1.000000000$
$\frac{2}{1} = 2.000000000$	$\frac{1}{2} = 0.500000000$
$\frac{3}{2} = 1.500000000$	$\frac{2}{3} = 0.666666667$
$\frac{5}{3} = 1.666666667$	$\frac{3}{5} = 0.600000000$
$\frac{8}{5} = 1.600000000$	$\frac{5}{8} = 0.625000000$
$\frac{13}{8} = 1.625000000$	$\frac{8}{13} = 0.615384615$
$\frac{21}{13} = 1.615384615$	$\frac{13}{21} = 0.619047619$
$\frac{34}{21} = 1.619047619$	$\frac{21}{34} = 0.617647059$
$\frac{55}{34} = 1.617647059$	$\frac{34}{55} = 0.618181818$
$\frac{89}{55} = 1.618181818$	$\frac{55}{89} = 0.617977528$
$\frac{144}{89} = 1.617977528$	$\frac{89}{144} = 0.618055556$
$\frac{233}{144} = 1.618055556$	$\frac{144}{233} = 0.618025751$
$\frac{377}{233} = 1.618025751$	$\frac{233}{377} = 0.618037135$
$\frac{610}{377} = 1.618037135$	$\frac{377}{610} = 0.618032787$
$\frac{987}{610} = 1.618032787$	$\frac{610}{987} = 0.618034448$

$$\frac{F_{n+1}}{F_n} = \frac{F_{n-1}}{F_n} + 1.$$

Fig. 8: The ratios of consecutive Fibonacci numbers.

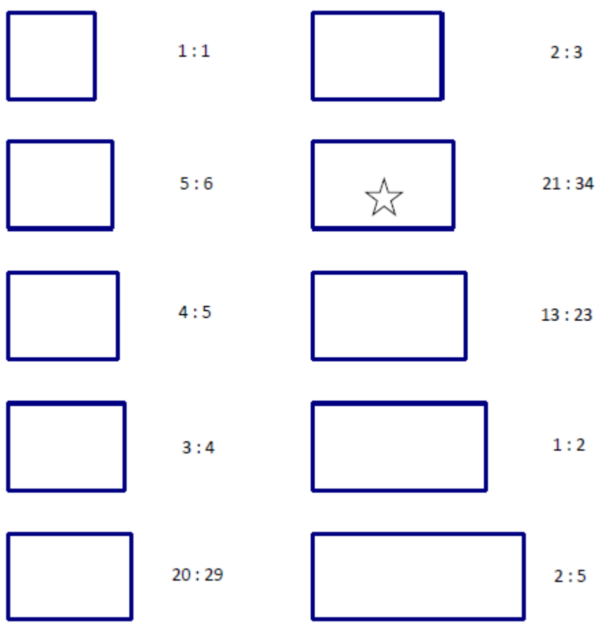

Fig. 9: Which rectangle is most pleasing to look at? Was this your choice?

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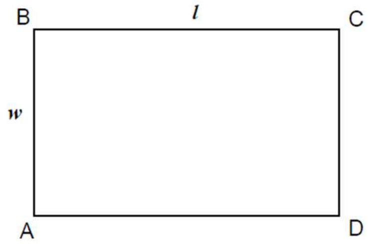
Perhaps the Golden Ratio gets its name from the fact that when a rectangle's sides are in the Golden Ratio, it is deemed to be the most attractive rectangle. Psychologists in the late 19th century often experimented to determine what the average person saw as the most attractive rectangle. You will notice above that the winner is the one of the star in it, and happens to be the one whose side lengths are Fibonacci numbers, which we saw above generates the Golden Ratio.

The Golden Ratio



$$\frac{width}{length} = \frac{length}{width + length}$$

$$\frac{length}{width} = \phi$$

$$\frac{1}{\phi} = \phi - 1$$

$$1 = \phi^2 - \phi$$

Then $\phi^2 - \phi - 1 = 0$

Applying the quadratic formula we get: $\phi = \frac{1 \pm \sqrt{5}}{2}$

$$\frac{1+\sqrt{5}}{2} = 1.6180339887498948482045868343656$$

Fig. 10: The Golden Ratio.

The Golden Ratio can also be seen on a regular pentagram, which is a 5-corner star generated from a regular pentagon.

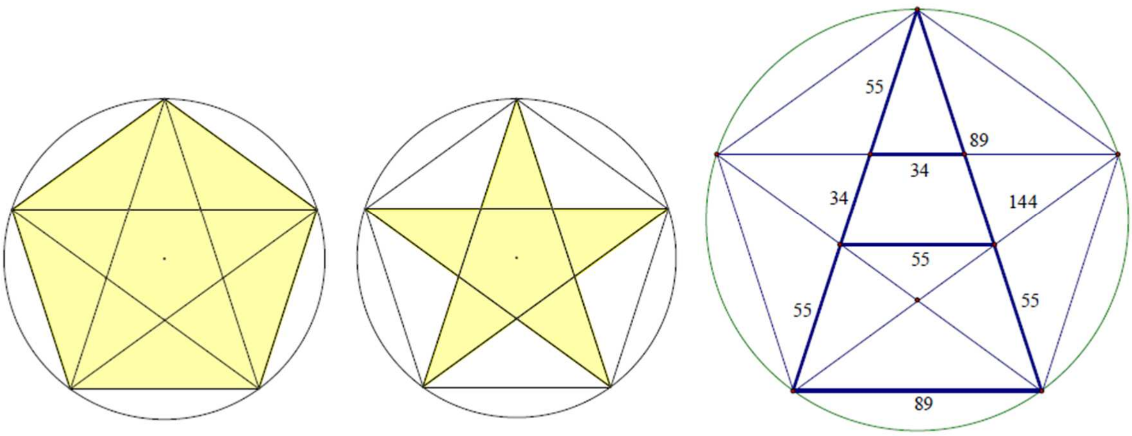


Fig. 11: The Golden Ratio and the regular pentagram.

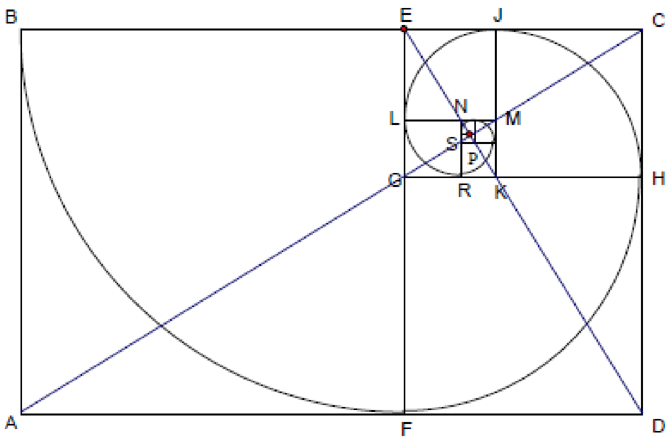


Fig. 12: Locating the vanishing point of the Golden Spiral.

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By consecutively cutting off squares from a Golden Rectangle, we find that as it gets smaller and smaller, there is a vanishing point, which is shown above, and at the same time it generates a spiral (figure 12). Which is then also seen in nature.

In architecture, there are numerous examples of the use of the Golden Ratio, as seen in the Parthenon in Athens, Greece, and in many cathedrals throughout the world, as shown below.

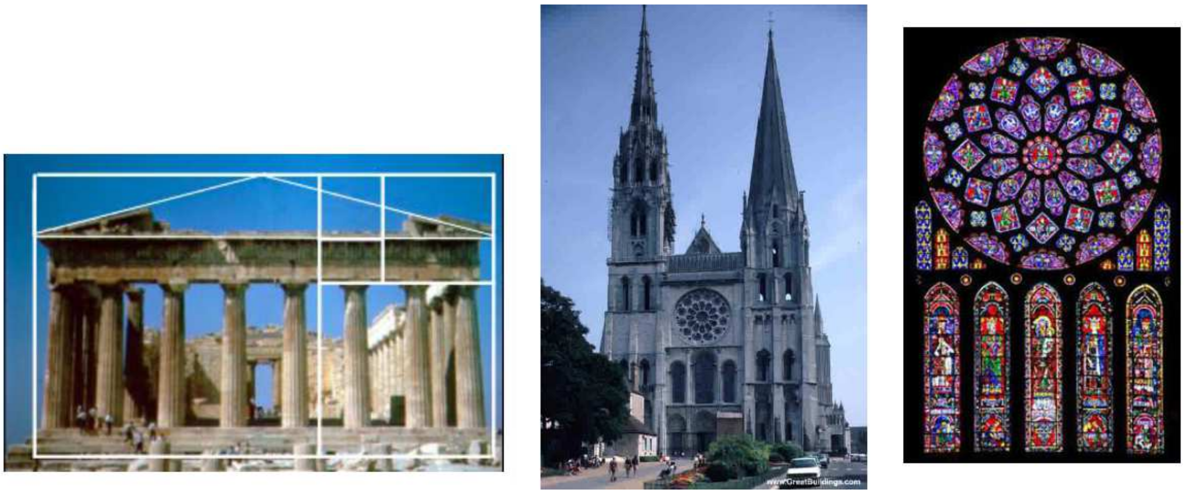

Fig. 13: The Golden Rectangle used in architecture.

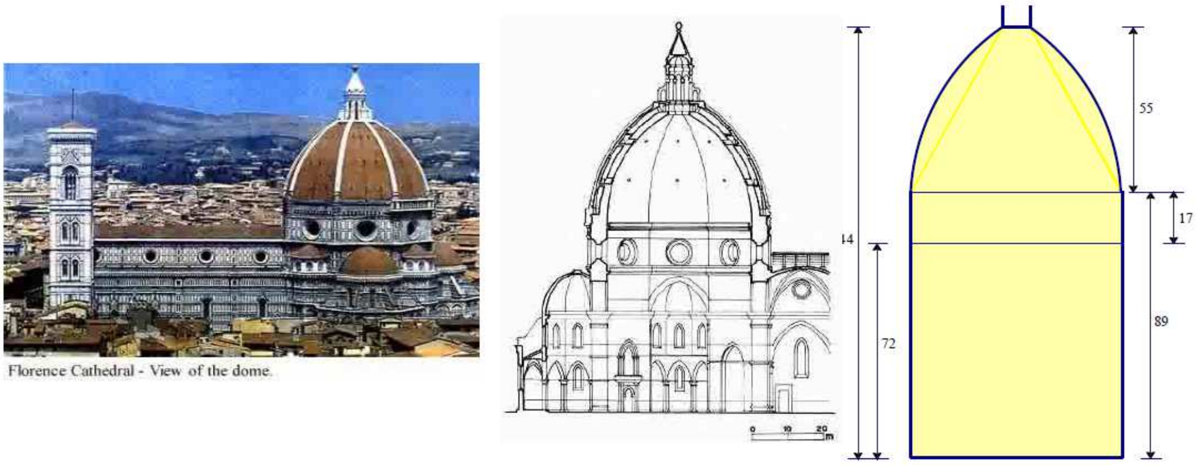

Fig. 14: Il Duomo di Firenze exhibits the Fibonacci numbers.

Fig. 15: Even the United Nations building in New York City has a shape approximating the Golden Rectangle.

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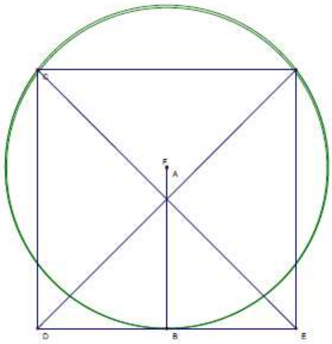
The famous Italian artist Leonardo da Vinci (1452 -1519) was clearly aware of the Golden Ratio. We see this evidence in the drawing he made of the Vitruvian man for a book on the very topic of the Golden Ratio, here referred to in Italian as the divine proportion, which is shown below.

$$\frac{\text{Side of square}}{\text{Radius of circle}} \approx \phi, \text{ which is the Golden Ratio}$$



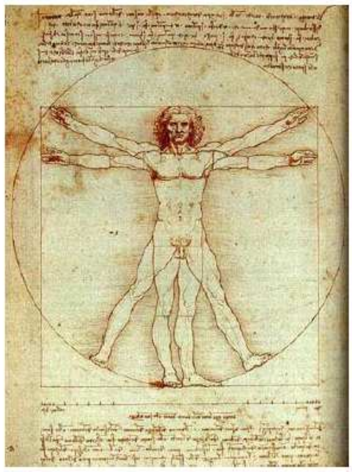


Fig. 16: The Vitruvian man.

Even as further evidence that Leonardo was aware of the Golden Rectangle, you can see below how he encased Mona Lisa's face in a Golden Rectangle. We also show that there are other portions of her picture that demonstrate the Golden Ratio.

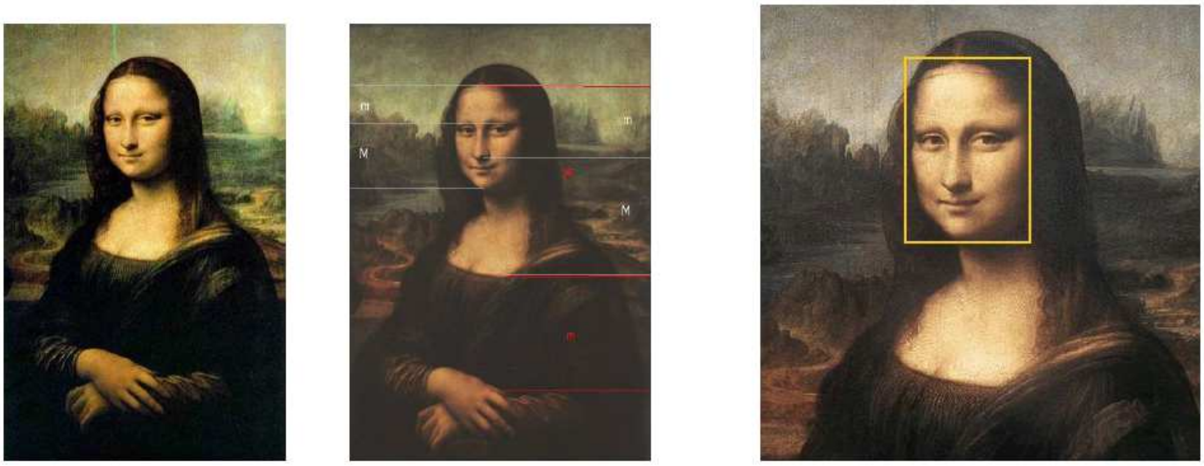


Fig. 17: The Golden Rectangle and Mona Lisa's face.

Countless artists have employed the Golden Ratio, as shown below.

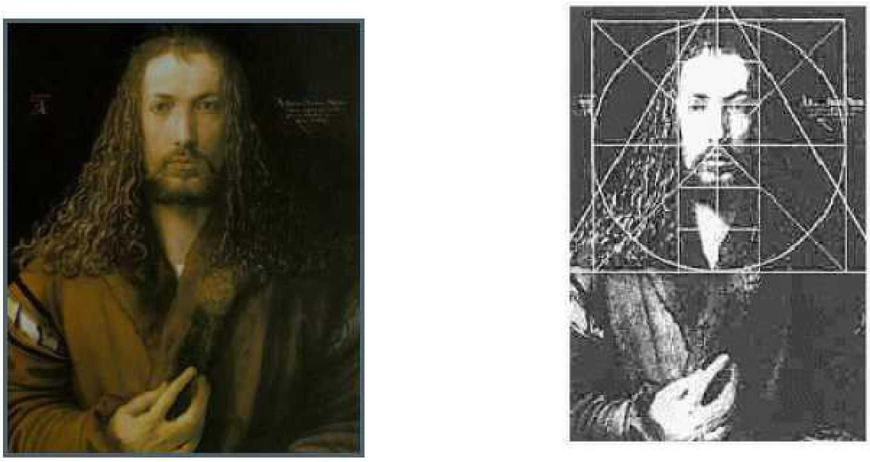


Fig. 18: Self-portrait of Albrecht Dürer.

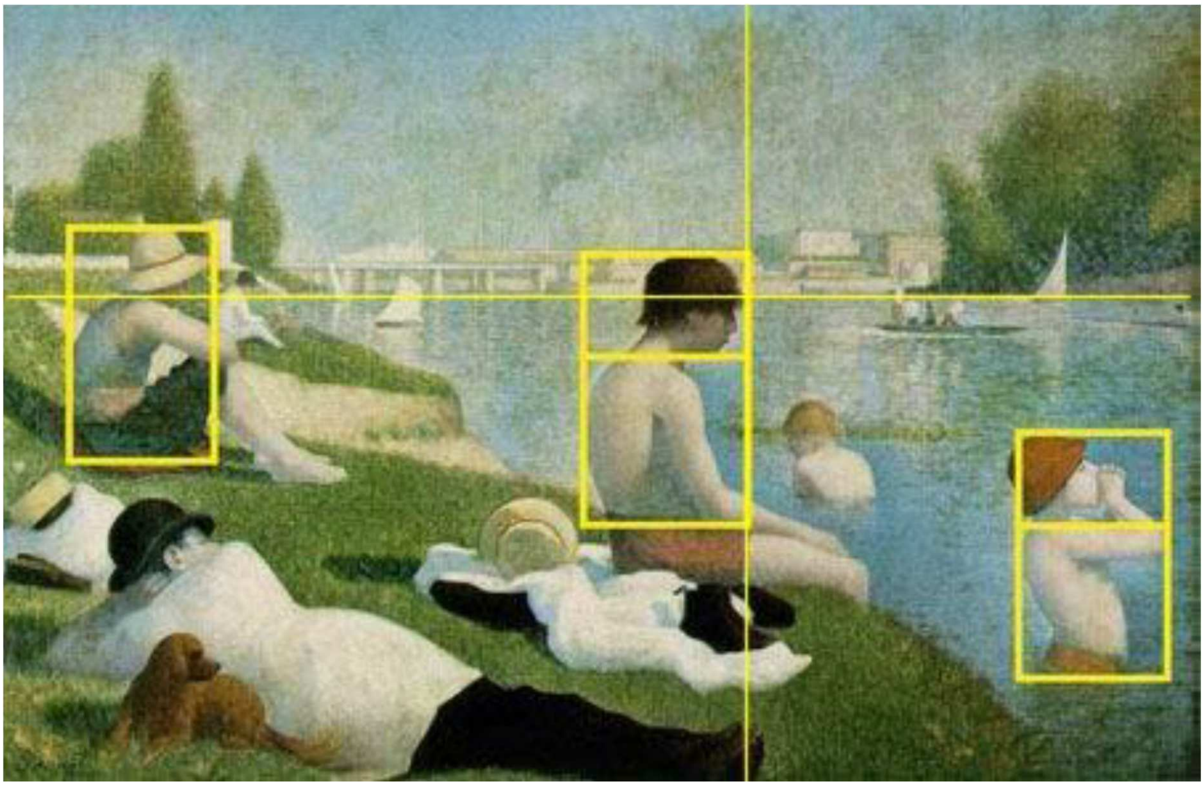


Fig. 19: *Bathers at Asmieres* by Georges Seurat.

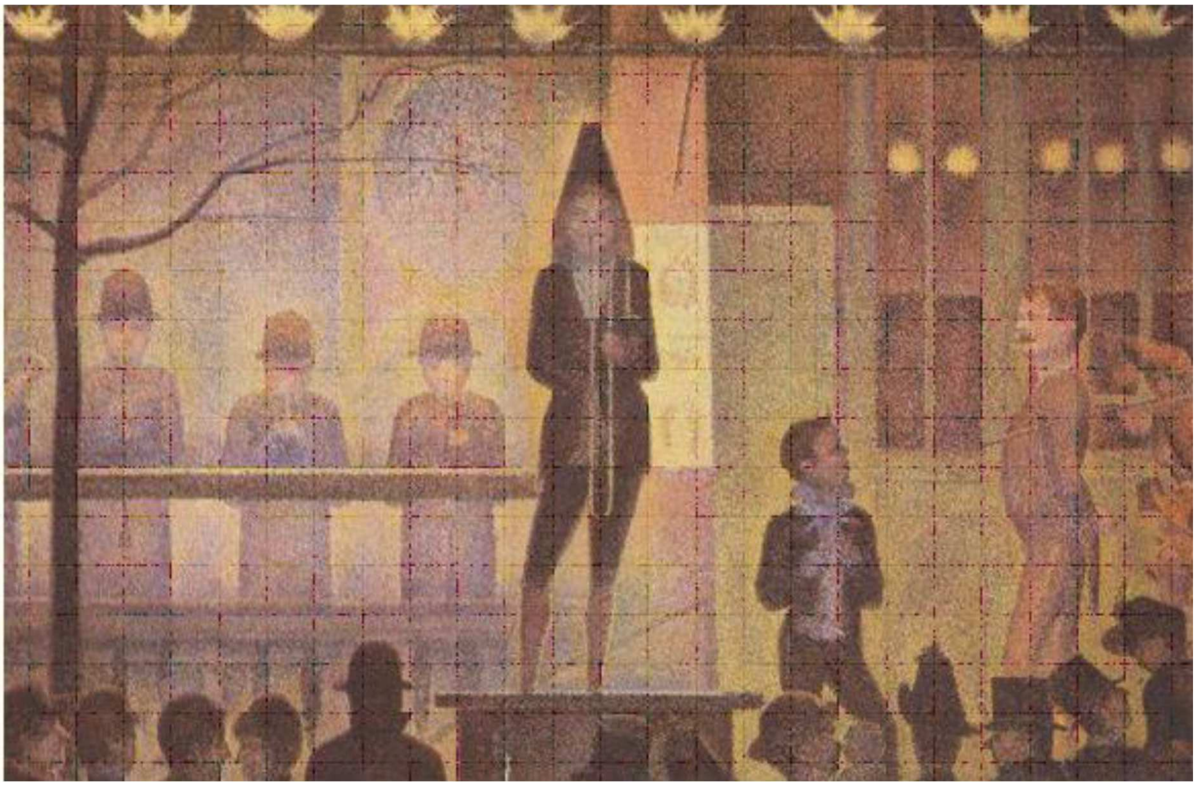


Fig. 20: The *Circus Parade* by Georges Seurat.

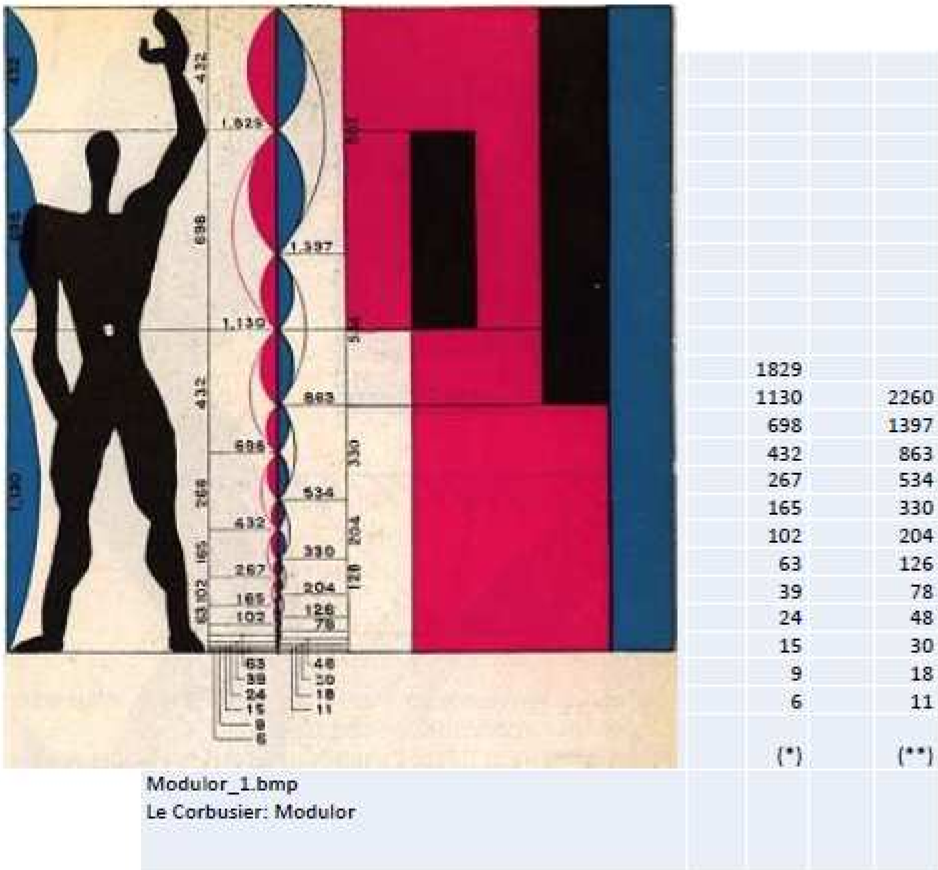


Fig. 21: *Modulor* by Le Corbusier.

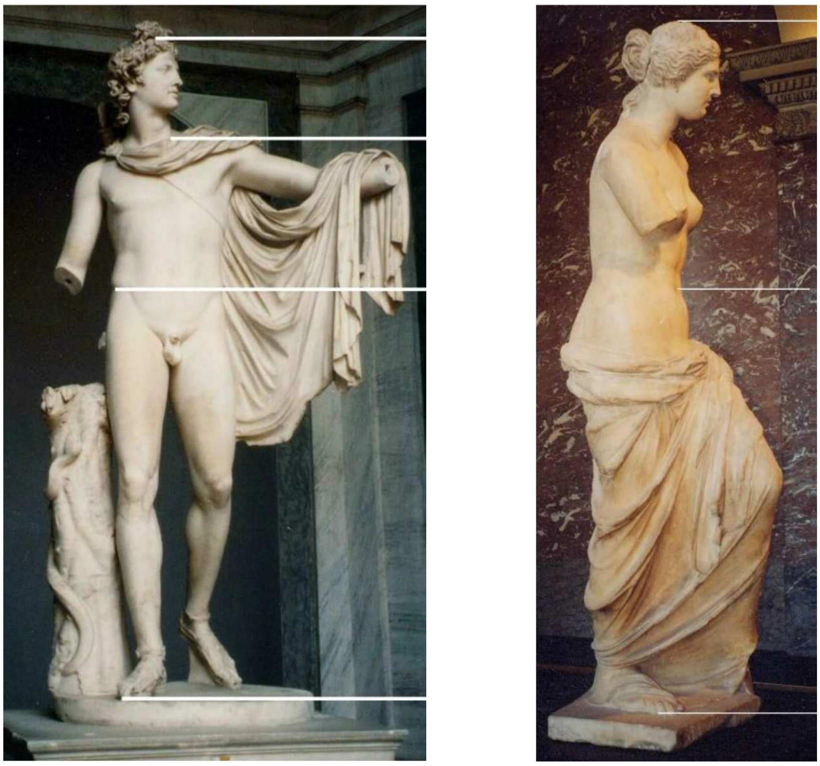


Fig. 22: Apollo Belvedere and Venus de Milos.

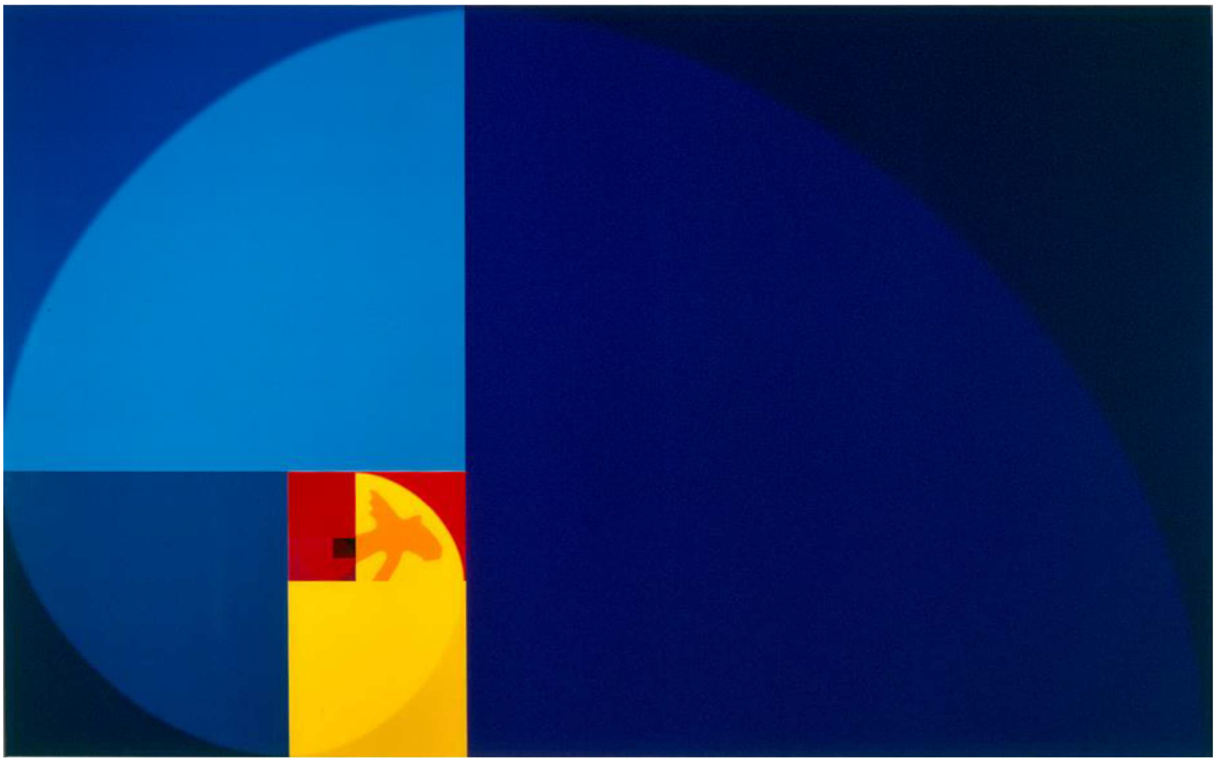


Fig. 23: The *Untitled* by Hreinn Fridfuinnsson.

3 The Magic Square

Albrecht Dürer (1471 -1528) also showed his penchant for mathematics in his famous etching "Melencolia I." You will notice in the upper right-hand corner there is a magic square, which is perhaps one of the most famous magic squares ever created. You may recall, that a magic square is a square arrangement of numbers whose rows, columns, and diagonals all have the same sum. What makes this magic square, even more amazing is that after a normal construction of a 4 x 4 magic square, Dürer interchanged the two center columns so that he could capture the year he made this picture in the year 1514 at the bottom cells of two center rows.

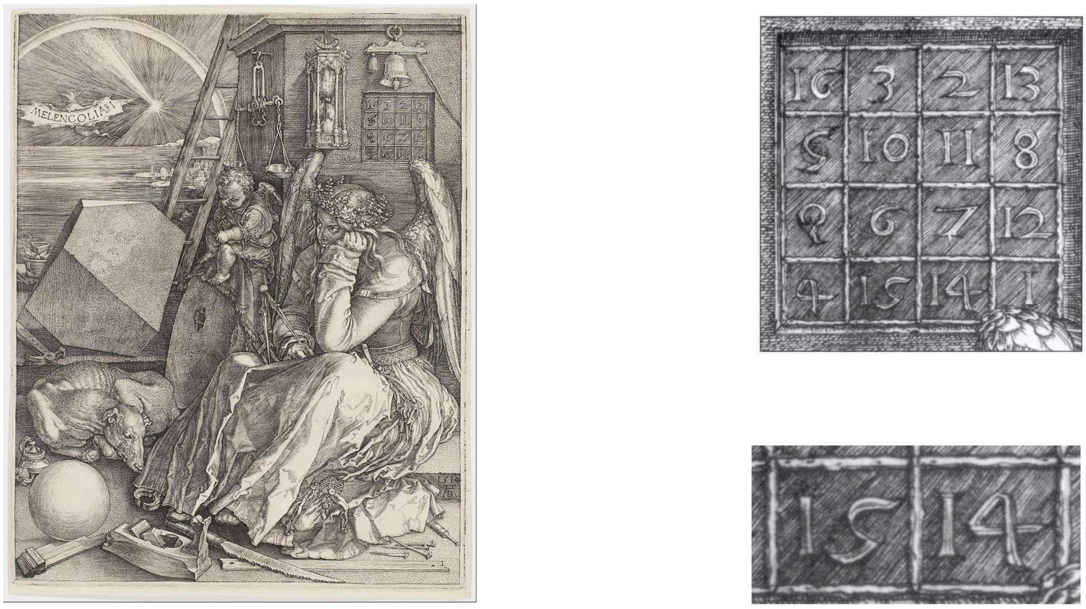


Fig. 24: *Melencolia I* by Albrecht Dürer and the magic square.

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Below you will see some of the unusual properties of this 4 x 4 magic square that are not shared by most other such magic squares.

Sum of all rows, columns and diagonals is 34

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

The four corner numbers have a sum of 34.
16 + 13 + 1 + 4 = 34

Each of the four corner 2 by 2 squares has a sum of 34.
16+3+5+10 = 34
2+13+11+8 = 34
9+ 6+ 4+15 = 34
7+12+14+1 = 34

The center 2 by 2 square has a sum of 34.
10 + 11 + 6 + 7 = 34

The sum of the numbers in the diagonal cells
equals the sum of the numbers in the cells not in the diagonals.
16+10+7+1+4+6+11+13 = 3+2+8+12+14+15+9+5 = 68.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

The sum of the squares of the numbers in the diagonal cells equals the sum of the squares of the numbers not in the diagonal cells.

$$16^2+10^2+7^2+1^2+4^2+6^2+11^2+13^2$$
$$= 3^2+2^2+8^2+12^2+14^2+15^2+9^2+5^2 = 748$$

The sum of the cubes of the numbers in the diagonal cells
equals the sum of the cubes of the numbers not in the diagonal cells.

$$16^3+10^3+7^3+1^3+4^3+6^3+11^3+13^3$$
$$= 3^3+2^3+8^3+12^3+14^3+15^3+9^3+5^3 = 9{,}248$$

The sum of the squares of the numbers in the diagonal cells
equals the sum of the squares of the numbers in the first and third rows.

$$16^2+10^2+7^2+1^2+4^2+6^2+11^2+13^2 = 16^2+3^2+2^2+13^2+9^2+6^2+7^2+12^2 = 748$$

Fig. 25: Properties of Dürer's magic square.

4 Perspectivity

We will now shift gears and consider the topic of perspectivity. We say that the segment AB and the segment DC are said to be in perspective at point P (which is their center of perspectivity). The term perspective was likely introduced from optics, since the eye placed at P would see the point D coinciding with the point A, and the point C coinciding with the point B.

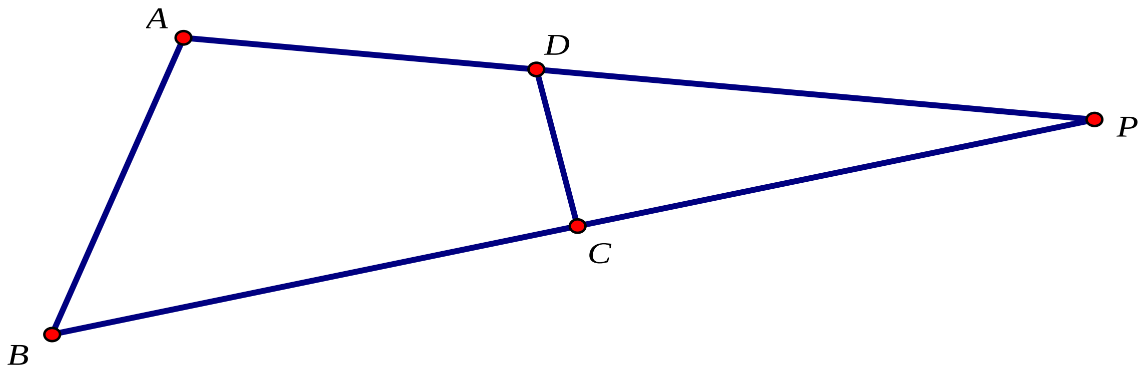

Fig. 26: Two segments in perspective.

Perhaps one of the best examples of perspectivity in art is Leonardo's "The Last Supper." Here you will notice how the many lines in the picture all seem to go to Jesus' is right eye. By using perspectivity throughout this picture Leonardo was the first artist to capture this theme with perfect depth perception.

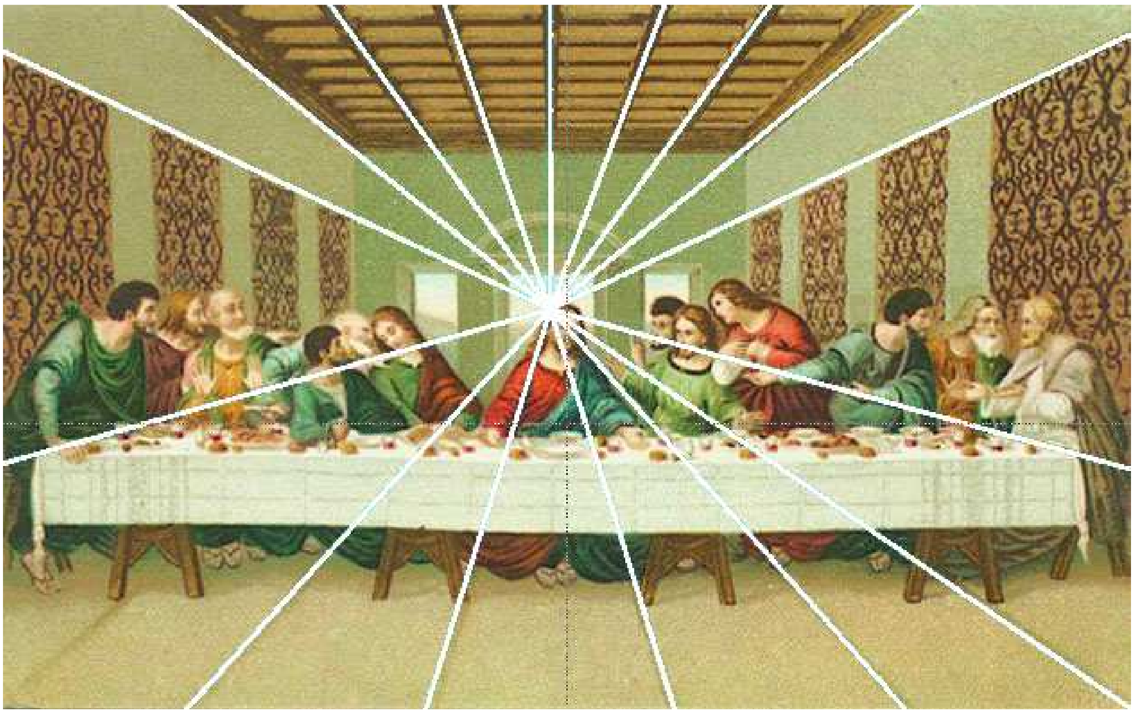

Fig. 27: *The Last Supper* by Leonardo da Vinci.

With this brief introduction into how mathematics is used in nature, art and architecture, it is hoped that the general public will look at the world, more critically, or shall we say, through a mathematical lens.

References

Posamentier, A. S., & Lehmann, I. (2007). *The Fabulous FIBONACCI Numbers.* Prometheus Books.
Posamentier, A. S., & Lehmann, I. (2012). *The Glorious Golden Ratio.* Prometheus Books.