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# The Transition to Enrichment in Mathematics Instruction: a Key Factor of Successful Teaching

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### *Abstract*

As more school systems tend to rate teachers by the performance of their students on tests, there is an ever-increasing tendency to "teach to the test." This brings a direct call for teachers of mathematics – one of the most tested topics in the curriculum – to develop a transition to enrichment topics. The intention of this investment of time is to make the students appreciate mathematics and to become more receptive learners. This paper will provide a series of ideas teachers can use to transition to an enrichment moment during their regular instruction with the hope that these few illustrations will guide the way to others that the teacher will find appropriate.


## Introduction

Why is mathematics the only subject in the school curriculum that adults are "proud" to have been weak in during their school years? One answer is that their teachers did not take the time to *transition* to enrichment topics in mathematics. What are the benefits of spending valuable class time to transition away from the required material, that is, from the standard lesson to enrichment topics? For one thing, through enrichment of mathematics, student interest and consequently generating enthusiastic learners should reap better results. This is further compounded by the teacher's "behavior" when presenting special topics in which the teacher also finds some wonderment – a characteristic that can often be favorably contagious. The transition to an enrichment topic can also open new horizons for the student and generate interest beyond the curriculum accompanied by some creative individual investigations – commonly seen as genuine learning! This investment of time taken from the standard curriculum tends to produce dividends in the form of a better, and more effective, learner of mathematics.

We can consider three types of enrichment approaches: acceleration, expansion and digression. The first is perhaps the least creative, in that the learner is simply moved along a faster track. It is the second two that we shall focus on here.

## Illustration 1

There are probably endless opportunities to digress from the study of arithmetic. One such might be to consider the origin of our place-value system and the numerals that we currently use. This allows us to consider Leonardo of Pisa, or more commonly known today as Fibonacci. We call them Hindu Arabic numerals, because in the first lines of the Introduction of his book *Liber Abaci* (1202), Fibonacci referred to the numerals 9,8,7,6,5,4,3,2,1 and the zefir (0) as Indian numerals that he encountered during his time on the Barbary Coast of Africa. This was the first time that these numerals appeared in Europe. One can then transition to the Fibonacci numbers – perhaps the most ubiquitous numbers in mathematics, and beyond. They stem from a problem on the regeneration of rabbits in chapter 12 of his book. These numbers: 1,1,2,3,5,8,13,21,34,55,89, 144, (where the sum of any two consecutive numbers determines the next number) are found in many areas from finance to biology, and more. These can fascinate students who just want to see their numeric curiosities,

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such as the sum of the first $n$ Fibonacci numbers is always 1 less than the $n+2^{nd}$ Fibonacci number (That is, $\sum_{1}^{n} F_n = F_{n+2} - 1$). The degree to which this transition should be taken is a function of the class level and the teacher's interest. Simply stated, it is boundless!

## Illustration 2

A class may be "caught up" in some arithmetic drill work. A nice transition might be to look at some unusual numbers and the patterns they may generate. Take for example, the strange number 666, which is often referred to as "the number of the beast" in many transcripts of chapter 13 of the New Testament's *Book of Revelation*. For starters, it is represented in Roman numerals – using all the numerals up to 100 in sequence: DCLXVI.

Also, 666 is equal to the sum of the first 36 natural numbers:

1+2+3+4+5+6+7+8+9+10+11+12+13+…+31+32+33+34+35+36 = 666.

Furthermore, 666 is the sum of the squares of the first seven prime numbers:

$$2^2 + 3^2 + 5^2 + 7^2 + 11^2 + 13^2 + 17^2 = 666$$

When students are asked to find other novelties of this unusual number they may come up with such things as:

The sum of the digits of 666 (6+6+6) is equal to the sum of the digits of its prime factors (666 = 2×3×3×37), namely, 6+6+6 = 18 = 2+3+3+3+7. Or the really creative may show how the numerals in sequence (1,2,3,4,5,6,7,8,9) can be arranged to get a sum of 666:

Such as 1+2+3+4+567+89=666, or 123+456+78+9.

They can be further creative by considering the numerals in reverse order to get the sum of 666:

Such as 9+87+6+543+21=666.

What appears to be a game, does give students some quantitative exercise that helps enrich their understanding of numbers.

Just for pure entertainment and also allowing students to see connections in mathematics – when they are least expected – one might show them that the first 36 decimal places of π has a sum of 666:

π =
3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535 9. . .

## Illustration 3

It is not uncommon for teachers in their effort to instill in students the "crime" of dividing by zero, and often tell them that the eleventh commandment ought to be "Thou shall not divide by zero." To make this point stick, a transition to a "proof" that has a ridiculous result could be of value. Provide the following – a step by step "proof" and see if students can find the error.

Let $a = b$

Multiply both sides by $a$: $a^2 = ab$

Subtract $b^2$ from both sides: $a^2 - b^2 = ab - b^2$

Factor both sides: $(a+b)(a-b) = b(a-b)$

Divide both sides by $(a-b)$: $a + b = b$

Since $a = b$: $2b = b$,

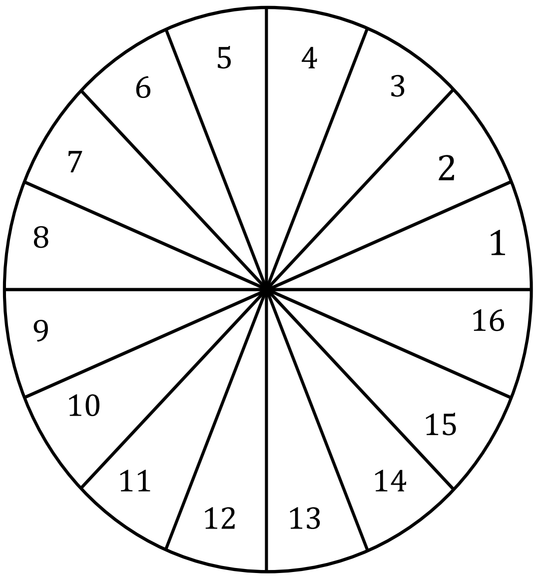
Then 2 = 1!

Their wonderment with this result should cause them to carefully inspect their "proof"- and search where the "crime" committed occurred. Eventually they should discover that we divided both sides of the equation by zero in form of *a - b.* This caused the ridiculous result.

## Illustration 4

The area of a circle is a common topic in geometry; yet, too rarely are students given a chance to truly appreciate the development of this formula. A transition of some sort of justification enriches the understanding of this popular formula.

One possible way to do this is to partition the circle into 16 (or more) sectors as show below.



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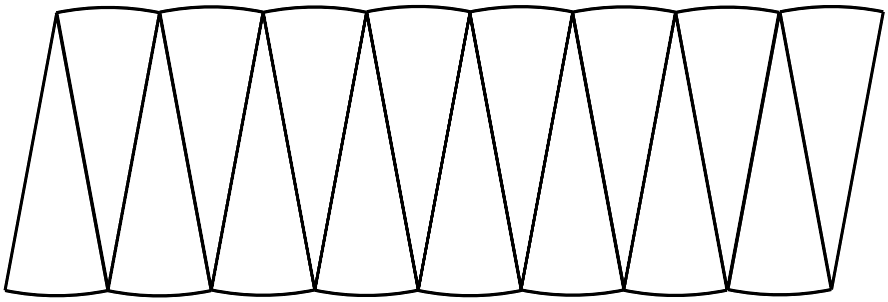


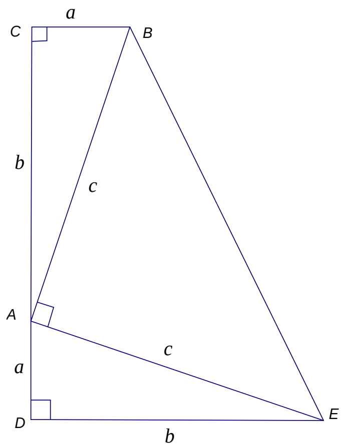
Then by rearranging the sectors to form what appears to be a parallelogram, the area can easily be found by taking the product of the height (which is the radius of the circle) and the base (which is half of the circumference of the circle to get

$$\left(\frac{1}{2}C\right)r = \frac{1}{2}(2\pi r)r = \pi r^2$$

## Illustration 5

The popular topic of the Pythagorean Theorem provides many opportunities to transition to some enrichment of this important relationship. There are books that provide a treasury of over 400 proofs of this theorem. Students could be asked what Pythagoras, Euclid and U.S. President James A. Garfield have in common. Yes, each has developed an original proof of the Pythagorean Theorem! Garfield's proof is particularly clever and uses some of the most basic concepts of mathematics. A lovely transition to a fruitful enrichment activity.
He partitioned a trapezoid into three right triangles (as shown below) and took the sum of the areas of these three triangles to get:

$$\frac{1}{2}ab+\frac{1}{2}ab+\frac{1}{2}c^2 = 2\left(\frac{1}{2}ab\right)+\frac{1}{2}c^2$$



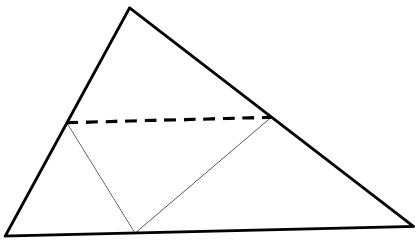

Then equating this sum to the common formula for the area of a trapezoid, he got:

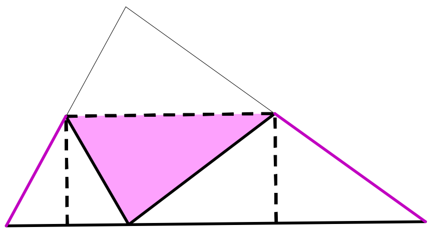
$$2\left(\frac{1}{2}ab\right)+\frac{1}{2}c^2 = \frac{1}{2}(a+b)^2$$

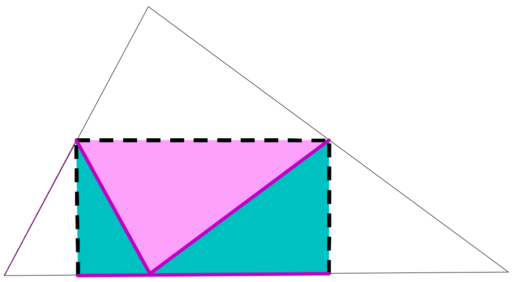
This then reduces to the familiar relationship: $a^2+b^2=c^2$

## Illustration 6

There are many ways to enrich geometry instruction with paper folding.  The Pythagorean Theorem can be demonstrated in this way, as can, for example, the sum of the angles of a triangle be shown to be 180° through paper folding.







## Illustration 7

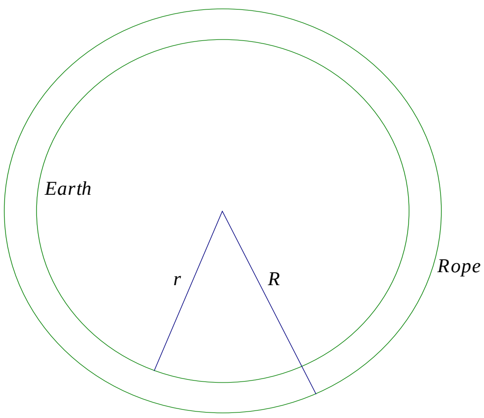
Throughout the teaching of mathematics we try to instill among the students the notion that we must be able to justify (or prove) any relationships before accepting them to be correct. Consider a transition to a demonstration that will demonstrate how a seeming-correct pattern need not necessarily be true.  Let's inspect the following statement: Every odd number greater than 1 can be expressed as the sum of a power of 2 and a prime number. Initially, this seems perfectly plausible.  When we begin to list the odd numbers: 3, 5, 7, 9, 11, and so on, we can see how they can be expressed in this way.  As we look down the list of these odd numbers (below), we notice that this relationship holds true quite a while, until we reach the odd number 127, where it no longer holds true.  This transition is very powerful in that it demonstrates for the learner that we cannot simply accept a patter that *seems* to work – without establishing some sort of proof or rational justification that it works for all cases.

$$3 = 2^0 + 2$$
$$5 = 2^1 + 3$$
$$7 = 2^2 + 3$$
$$9 = 2^2 + 5$$
$$11 = 2^3 + 3$$
$$13 = 2^3 + 5$$
$$15 = 2^3 + 7$$
$$17 = 2^2 + 13$$
$$19 = 2^4 + 3$$
$$\vdots$$
$$51 = 2^5 + 19$$
$$\vdots$$
$$125 = 2^6 + 61$$
$$127 = ?$$
$$129 = 2^5 + 97$$
$$131 = 2^7 + 3$$

## Illustration 8

Although much of the transitions to enrichment in mathematics might seem to be entertaining – as well they ought to be – they all carry with them an important message, one that gives some true "life" to the subject. One important lesson to be learned as part of mathematics instruction is that intuition cannot always be trusted, and we must be able to inspect relationships analytically.

Take for example, the following counterintuitive situation: Suppose a rope is tied tightly around the Earth along the equator – at a length of about 24,000 miles. If we lengthen the rope by just one yard and uniformly loosen the rope around the Earth, would a mouse fit beneath the rope? Our intuition would clearly tell us, no. However, analyzing the situation described leads us to the following:




The circumferences of the two concentric circles can be represented as follows:

$$C = 2\pi r, \quad \text{or} \quad r = \frac{C}{2\pi}$$

and

$$C + 1 = 2\pi R, \quad \text{or} \quad R = \frac{C+1}{2\pi}$$

We need to find the difference of the radii of the two concentric circles:

$$R - r = \frac{C+1}{2\pi} - \frac{C}{2\pi} = \frac{1}{2\pi} \approx .159 \text{ yards} \approx 5.7 \text{ inches}$$

Clearly, a mouse can crawl under the rope, which, amazingly, is now over 5½ inches above the surface of the Earth! Such digressions can open new sorts of thinking in the learner, which is an important aspect in learning mathematics.

## Conclusion

Transitions that truly enrich mathematics instruction are particularly important today, when there is a universal tendency to test students with many related aspects that affect the teaching process. In the United States today there are many states that evaluate teacher effectiveness by their students' results. This drives many teachers to "teach to the test" with no time allocated for any enrichment of the subject. Although in the eyes of many teachers, they are maximizing student performance on ensuing tests, they are also sacrificing the opportunity to transition to enrichment and thereby shortchanging this all importance subject. These transitions have at least two very important effects: They are an investment of time that should lead students to be more receptive learners for future mathematics topics, and they will leave students with a better feeling about mathematics – a subject that most adults today seem to be proud of not liking, and not having done well in during their school days.

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