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Nspire CAS and Laplace Transforms

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Overview

- Introduction
- Two Specific Applications
  - Mass-spring problem
  - RLC circuit problem
- Piecewise and Impulse Inputs
- Using the convolution

Introduction

- There is no built-in Laplace transform function in Nspire CAS.
- But we can download an Nspire CAS library for using this stuff.
- For details about the Laplace transforms library "ETS_specfunc.tns", see the document of Chantal Trottier:

http://seg-apps.etsmtl.ca/nspire/documents/transf%20Laplace%20prog.pdf.

Introduction

- In this talk, we will use this Laplace transforms package to automate some engineering applications, such as mass-spring problem and RLC circuit.
- We will also use an animation: this helps students to get a better understanding of what is a "Dirac delta function".
- Also, the convolution of the impulse response and the input will be used to find the output in a RLC circuit with various input sources.

Introduction

- This is the pedagogical approach we have been using at ETS in the past 3 years with Nspire CAS CX handheld: using computer algebra to define *functions*. And encouraging students to define their own functions.
- These functions are saved into the library Kit_ETS_MB and can be downloaded from the webpage at: https://cours.etsmtl.ca/seg/MBEAUDIN/

Introduction

- The rest of this short Power Point file gives an overview of what will be done live, using Nspire CAS.

Two Specific Applications

- A damped mass-spring oscillator consists of an object of mass m attached to a spring fixed at one end.
- Applying Newton's second law and Hooke's law (let k denote the spring constant), adding some friction proportional to velocity (let b the factor) and an external force $f(t)$, we will obtain a differential equation.

Two Specific Applications

- One can show that the position $y(t)$ of the object satisfies the ODE

$$m\,y''(t) + b\,y'(t) + k\,y(t) = f(t), \quad y(0) = y_0,\ y'(0) = v_0.$$

- Students are solving this problem using numeric values of the parameters m, b and k and various external forces $f(t)$. The Laplace transform is applied to both sides of the ODE and, then, the inverse Laplace transform.

Two Specific Applications

- "Laplace transforms tables" are included in their textbook: students use the different properties to transform the ODE into an algebraic equation involving Y(s) (the Laplace transform of $y(t)$).
- The CAS handheld is used for partial fraction expansion. Term by term inverse transforms are found using the table.

Two Specific Applications

- The library ETS_specfunc is used to check their answers.
- This is how we have been proceeding at ETS since many years (TI-92 Plus, V200): for more details, see http://luciole.ca/gilles/conf/TIME-2010-Picard-Trottier-D014.pdf

Two Specific Applications

- Now we want to automate this process. Having gained confidence with Laplace transforms techniques, we solve *by hand* ***without*** *numeric values* the ODE

$$m\,y''(t)+b\,y'(t)+k\,y(t)=f(t),\quad y(0)=y_0,\,y'(0)=v_0.$$

- We find the following solution:

$$y(t)=\mathrm{Ilap}\left(\frac{msy_0+mv_0+by_0+\mathrm{Lap}(f)}{ms^2+bs+k}\right)$$

Two Specific Applications

- Why not define a "mass-spring" *function*? In French, "spring" is "ressort":

This is a *function* of 6 variables

$$\text{Ilap}\left(\frac{m \cdot s \cdot yo + m \cdot vo + b \cdot yo + \text{Lap}(f)}{m \cdot s^2 + b \cdot s + k}\right) \rightarrow \text{ressort}(m, b, k, f, yo, vo)$$

- This is done in Nspire CAS using the "laplace" and "ilaplace" functions defined in the library ETS_specfunc (variables are necessary "*t*" and "*s*").

Two Specific Applications

- The same procedure can be applied to a series RLC circuit: we consider a voltage source $E(t)$, a resistor R, an inductor L and a capacitor C.
- In this case, Kirchhoff's voltage law (also Ohm's law and Faraday's law) are used to construct a model.

Two Specific Applications

- Textbooks give the details and the following ODE for the voltage across the capacitor is obtained:

$$LC\,v_C''(t) + RC\,v_C'(t) + v_C = E(t), \quad v_C(0) = v_{c0}, i(0) = i_0.$$

- Here $i(t)$ is the current at time t. The relation is given by

$$v_C'(t) = i(t)/C$$

Two Specific Applications

- The similitude with the former ODE can be exploited.

$$m\,y''(t) + b\,y'(t) + k\,y(t) = f(t), \quad y(0) = y_0,\ y'(0) = v_0.$$

$$LC\,v_C''(t) + RC\,v_C'(t) + 1\,v_C = E(t), \quad v_C(0) = v_{c0}, i(0) = i_0.$$

- So we can define a single function that solves this problem:

$$\text{ressort}\left(L \cdot C, R \cdot C, 1, E, vco, \frac{io}{C}\right) \rightarrow \text{circuit}\left(R, L, C, E, vco, io\right)$$

Another *function* : of course,the order of the variables could have been different.

Piecewise and Impulse Inputs

- Main reason why students are introduced to Laplace transforms techniques: to be able to consider piecewise external forces (or voltage sources).
- In order to do this, we first need to define the unit-step function $u(t)$. Then we can define the rectangular pulse $p(t)$.

Piecewise and Impulse Inputs

$$u(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases}, \quad p(t) = u(t) - u(t - T)$$

- Then, we can define the unit-impulse (or Dirac delta) function $\delta(t)$ as follow:

$\delta(t) = 0$ for $t \neq 0$

$\delta(t)$ is undefined for $t = 0$

$$\int_{t_1}^{t_2} \delta(t)\, dt = \begin{cases} 1, & t_1 < 0 < t_2 \\ 0, & \text{otherwise} \end{cases}$$

Piecewise and Impulse Inputs

- Engineering students don’t need to be introduced to generalized functions.
- So instead of saying that the unit impulse is the derivative of the unit step function, we can use limiting arguments for a good understanding of this particular “function”.
- Here are the details.

Piecewise and Impulse Inputs

- Let a be a non negative fixed number. Let $\varepsilon > 0$. Use the rectangular pulse function

$$f_\varepsilon(t) = \begin{cases} 0, & t < a \\ \dfrac{1}{\varepsilon}, & a < t < a + \varepsilon \\ 0, & t > a + \varepsilon \end{cases}$$

- Note that this is a scaled indicator function of the interval $a < t < a + \varepsilon$.

Piecewise and Impulse Inputs

- A good method to really understand what is the meaning of the Dirac delta function would be to use a limiting process.
- Example: we will consider

$$y'' + 4y' + 8y = 50\delta(t-3),\ y(0) = 2,\ y'(0) = 0.$$

- We will solve this directly using the "ressort" function defined earlier.

Piecewise and Impulse Inputs

- But we will also solve the ODE

$$y'' + 4y' + 8y = \frac{50}{\varepsilon}\left(u(t-3) - u(t-3-\varepsilon)\right),\ y(0) = 2,\ y'(0) = 0.$$

- Then, we will animate the solution, starting with $\varepsilon = 1$ and getting closer to 0.
- This is, in fact, the main idea behind an *impulse function*.

Using the convolution

- Finally, consider the ODE

$$ay''(t) + by'(t) + cy(t) = x(t)\,u(t), \qquad y(0) = 0, y'(0) = 0$$

- Let $h(t)$ be the inverse Laplace transform of the so called transfer function:

$$\frac{1}{as^2 + bs + c}$$

- Then the solution of the ODE is given by

$$y(t) = x(t) * h(t) \equiv \int_0^t x(\tau)h(t-\tau)\,d\tau$$

Using the convolution

- A word about the integral

$$y(t) = x(t) * h(t) \equiv \int_0^t x(\tau) h(t-\tau)\, d\tau$$

- This is called the *convolution* of the input $x(t)$ with the impulse response $h(t)$. This impulse response is entirely determined by the components of the system.

Using the convolution

- Fortunately, in the case of Laplace transforms, we don't have to compute the integral in order to find the convolution of two signals $x(t)$ and $h(t)$.
- Instead, we use the "convolution property":

$$x(t) * h(t) = \text{Ilap}\left(\text{Lap}(x(t)) \cdot \text{Lap}(h(t))\right)$$

Using the convolution

- We will use this fact to find the output in a RLC circuit with various voltage sources.

- *Linearity* and *time invariance* will be illustrated (notion of a "LTI system").

Using the convolution

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Using the convolution

- Now, let's switch to Nspire CAS and show the examples to conclude this talk.

Thank You!