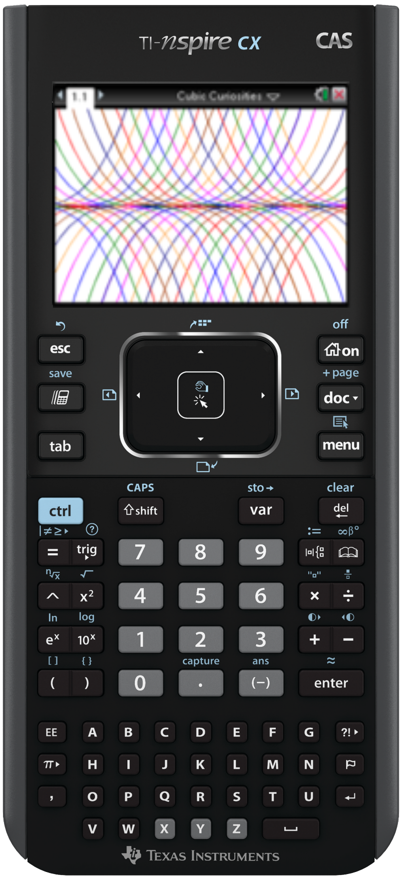
Cubic Curiosities

Koen Stulens
Guido Herweyers

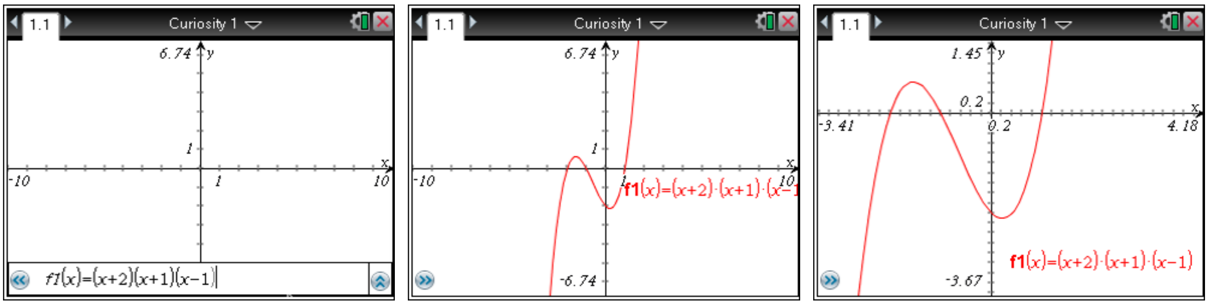


Curiosity 1 – Cubic Tangent Line

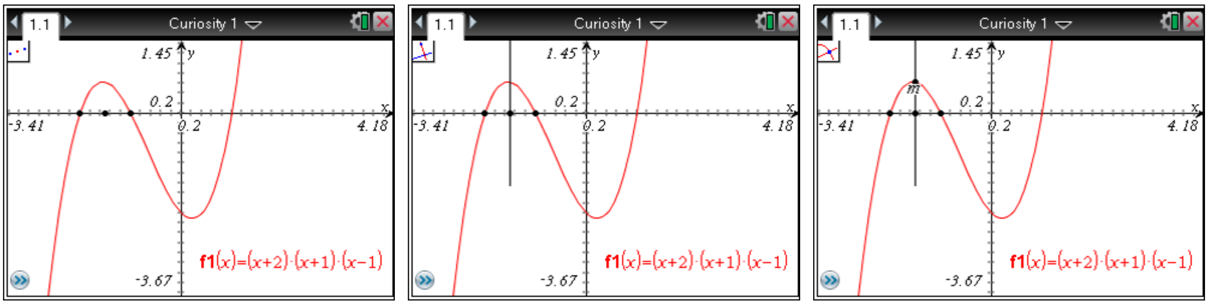
Based on *Information Technology in the Math Class*

J. Deprez, L. Gheysens, G. Herweyers, K. Stulens, .J Van Hee - T ***[3]*** ***Flanders***

Define – in Graphs & Geometry – the function $f1(x)=(x+2)(x+1)(x-1)$.

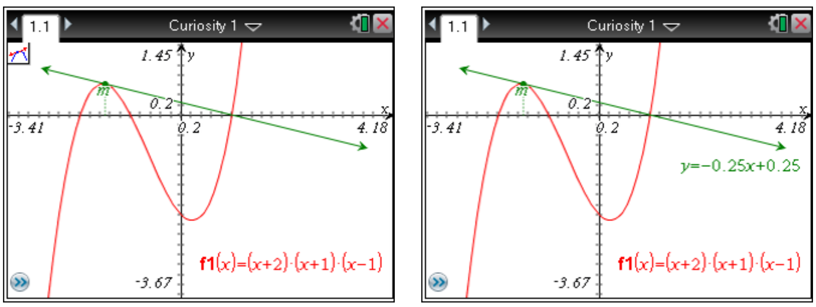


Determine, using the geometric Construction tools, the midpoint of $x=-2$ and $x=-1$; $x=-1.5$.
Draw the perpendicular to the x-axis at the point $x=-1.5$ and it's intersection point m with the graph of $f1$.



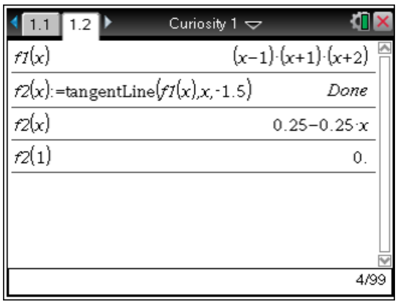
Plot, using the geometric tools Points & Lines, the tangent line at the midpoint of $x=-1.5$ to the graph of $f1$. Enlarge the tangent line by grabbing its begin and end point.

Together with some layout changes we get the following result. Conclusion (related to $x=1$)?
Right click on the tangent line ([CTRL] + [menu]) to get the equation of the tangent line.
Does the point (1,0) fulfill this condition?



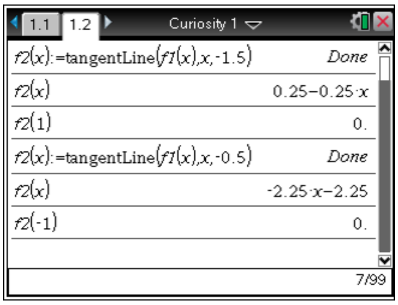
A similar investigation can be done via the Calculator:

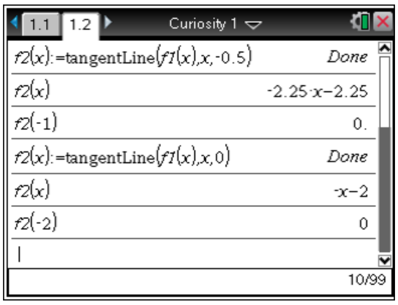
- First show the definition of $f1$ – defined in Graphs & Geometry – in the Calculator,
- Then define $f2(x) := \text{tangentLine}(f1(x), x, -1.5)$ (or "$\text{Define } f2(x) = \text{tangentLine}(f1(x), x, -1.5)$"),
- Calculate the function value of $f2$ in $x = 1$.



Is this pure coincidence or are we running into a property?

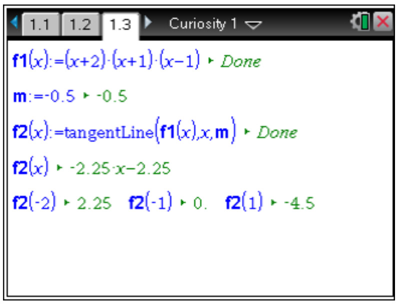
Check if our conclusion also counts for the other pairs of zeros: the tangent line at $x = 0$ between $x = -1$ & $x = 1$, and the tangent line at $x = -0.5$ between $x = -2$ & $x = 1$.

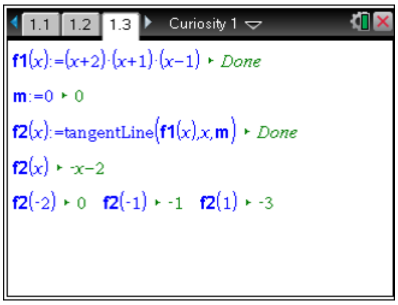


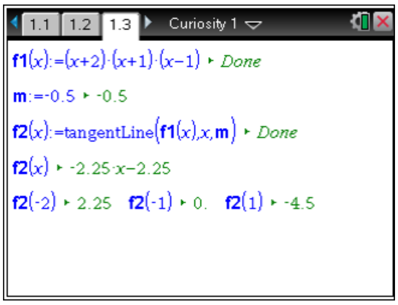


Note that we need to execute the commands in the Calculator each time again for any situation. We can avoid this to make the calculations in Notes using interactive Math Boxes.

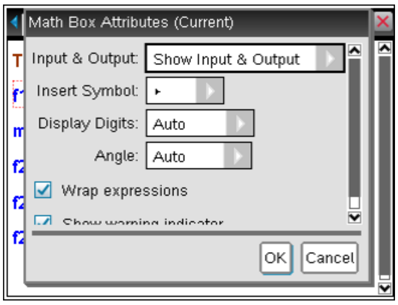
In Notes we only have to change the value of m to check the property for the different midpoints.

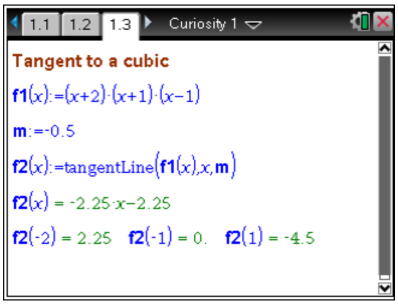






Right clicking on a Math Box let you change its format which can result in (some text added as well):





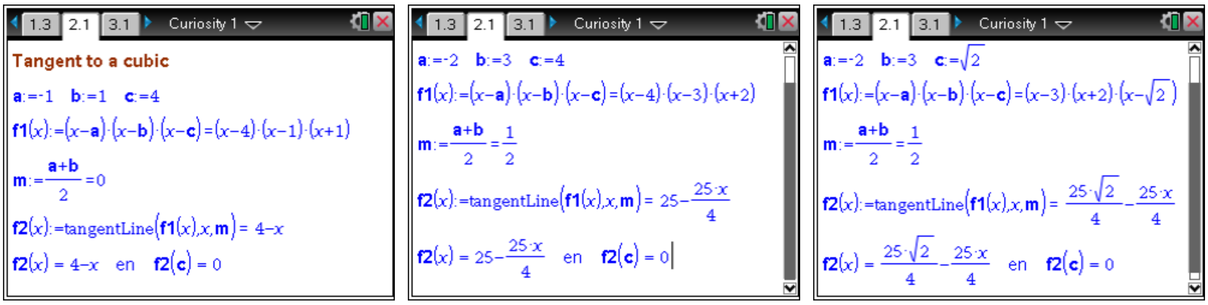
Does this conjecture depend on the chosen cubic function or does it leads us to the following property:

> *For a cubic function with three real roots, the tangent to the cubic $f(x)$ at the average of two roots intersects the graph of $f(x)$ at the third root.*

To investigate this property for some more cubic functions we use Notes for an interactive algebraic exploration and Graphs & Geometry for an interactive visualization.

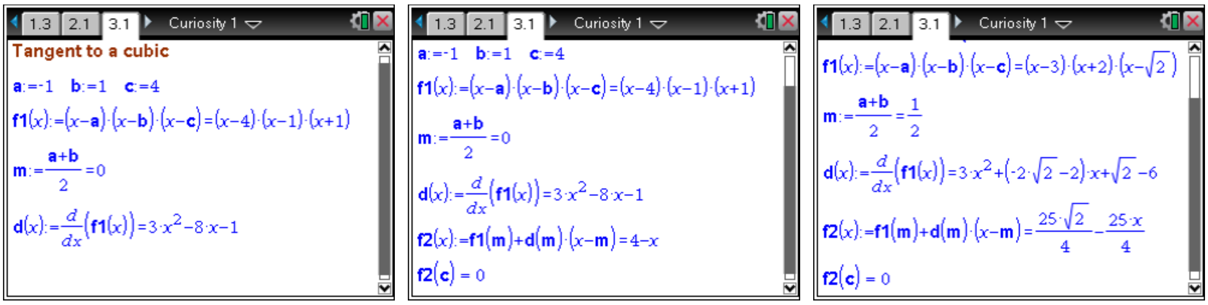
Notes

Let's define the roots of the cubic as variables a, b, c and the cubic as $f1(x) := (x-a)(x-b)(x-c)$.
To determine the tangent line you can use directly the tangentLine command



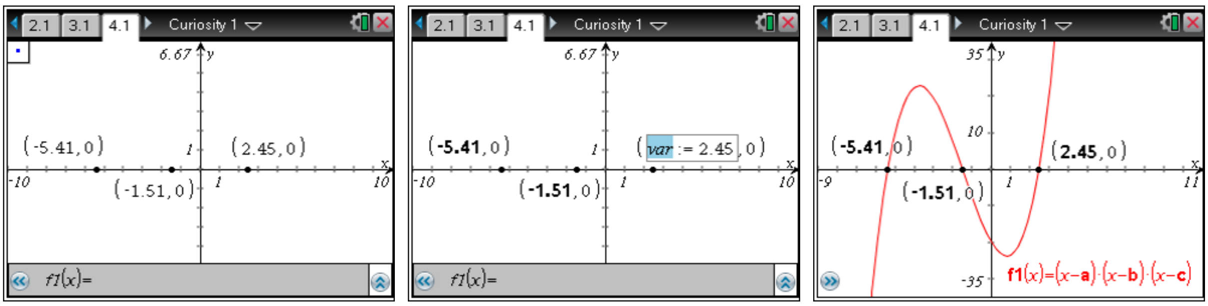
or the definition of tangent line based on the first derivative:

$a := -2, b = 3, c = \sqrt{2}$

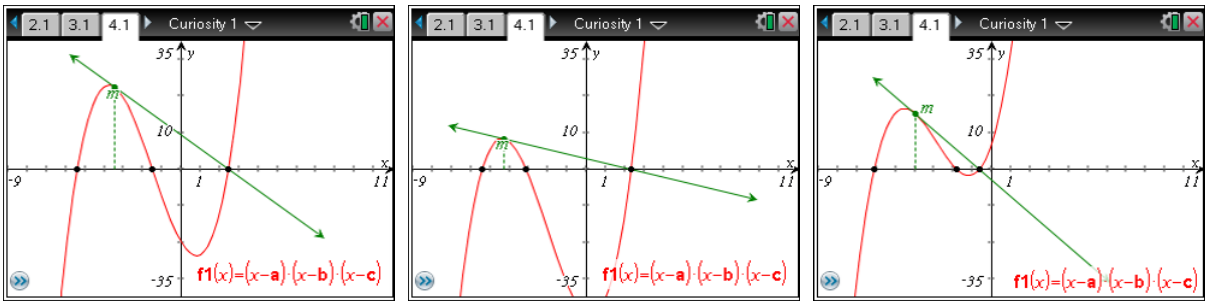


Graphs & Geometry

Drop three points on the x-axis and show their coordinates. Store the x-coordinates of these points as the variable a, b, c and define $f1(x) := (x-a)(x-b)(x-c)$.



Draw the tangent line at the midpoint of two roots similar as in the beginning of this activity.
To investigate if a generalization is acceptable, grab and move the points on the x-axis.



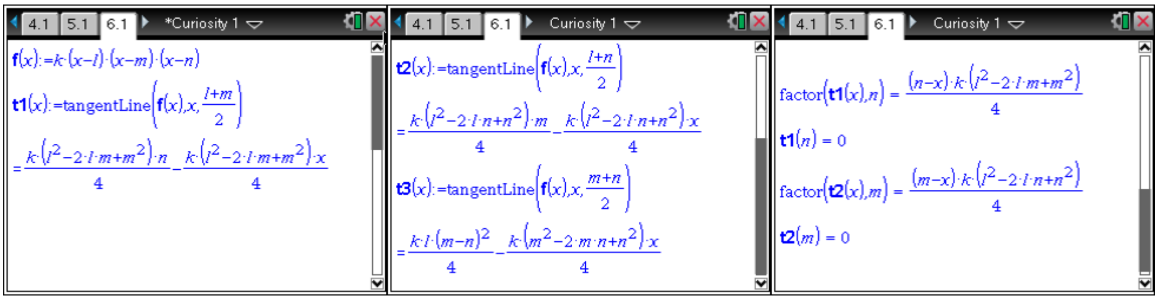
To show that the property is absolutely true we need to proof it for each arbitrary polynomial.

Therefore define $f(x):=k(x-l)(x-m)(x-n)$ with $k,l,m,n\in$ ° .

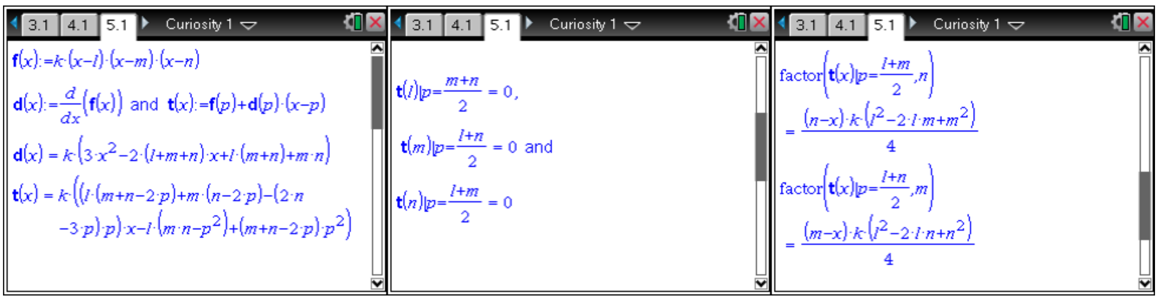
Because l,m,n are arbitrary numbers it's enough to proof that the tangent line at the graph of $f(x):=k(x-l)(x-m)(x-n)$ in the point $x=\dfrac{l+m}{2}$ intersects the graph of $f(x)$ at $x=n$.

CAS can be a handy assistant here for our reasoning. This can be done with:

- The tangentLine$(f(x),x,\dfrac{l+m}{2})$ command



- The definition of the tangent line at $x=\dfrac{l+m}{2}$: $y=f\left(\dfrac{l+m}{2}\right)+f'\left(\dfrac{l+m}{2}\right)\left(x-\dfrac{l+m}{2}\right)$.



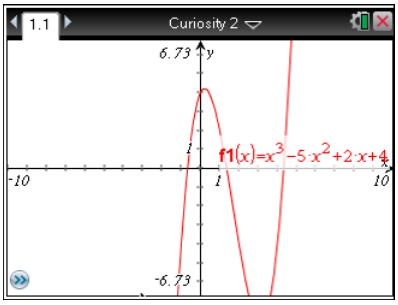
Curiosity 2 – Cubic Zeros

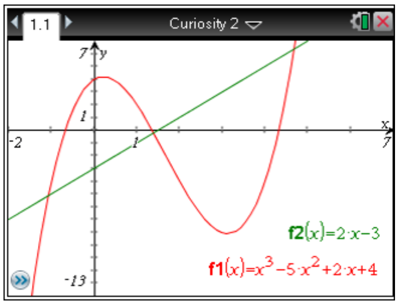
Based on *You must still think yourself, CAS can only help*

Gert Schomacker - T[3] Denmark – The Case for CAS - www.t3ww.org/cas

Classroom Activity

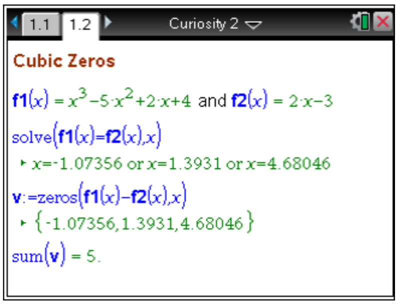
- Plot the graph of the cubic $f1(x) = x^3 - 5x^2 + 2x + 4$.
- Change the window settings to have a good view.
- Define a linear function $f2(x) = px + q$ that intersect the graph three times; where each student choose arbitrary values for p, q.





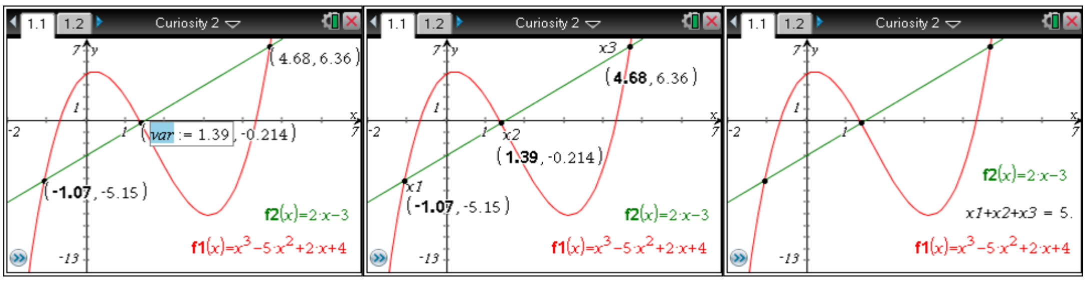
- Determine the intersections points of the line and the graph and calculate the sum $x1 + x2 + x3$ of their x-coordinates $x1, x2, x3$.

  ○ *Algebraically*



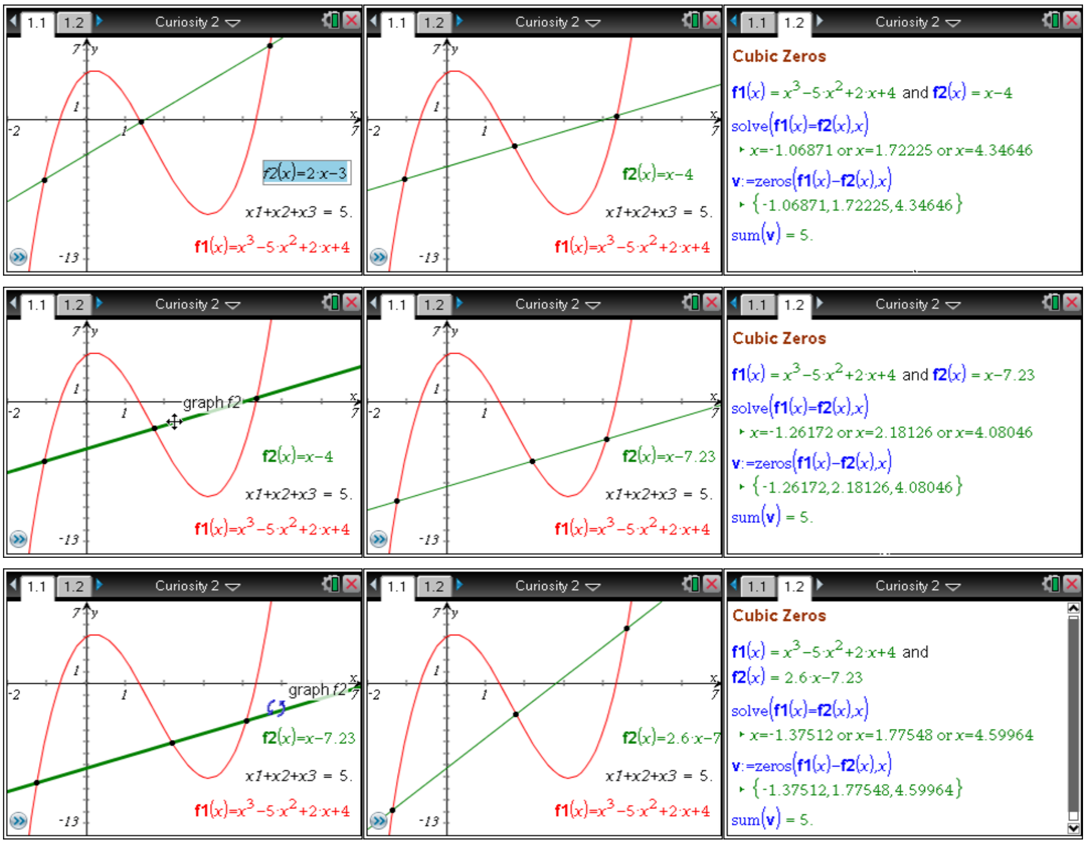
  ○ Graphically

  We store the x-coordinates as variables $x1, x2, x3$ and then perform a calculation on the text box $x1 + x2 + x3$.



It will be definitely surprising to the students that even though they have different values for the intersection points $x1, x2, x3$ they will all get the same sum $x1 + x2 + x3 = 5$.

Change the chosen line by changing the function definition in the text box or just grab and move the line.



Coincidence of regularity? Repeat the activity above for another cubic.

Conjecture

For any cubic $f(x) = x^3 + ax^2 + bx + c$ and any line $g(x) = px + q$ intersecting the graph of the cubic at three points, the sum of the *x*-values of the intersection points is $-a$.

Proofing the conjecture can be done with CAS with or without reasoning

- *Without reasoning*
  For the solve or zeros command you get pretty complicated expressions but the sum of the three solutions/zeros results into $-a$.

Cubic Zeros

$f(x):=x^3+a\cdot x^2+b\cdot x+c$

$g(x):=p\cdot x+q$

solve(f(x)=g(x),x) ▸

$$x=\frac{2\cdot\sqrt{a^2-3\cdot(b-p)}\cdot\cos\left(\frac{\sin^{-1}\left(\frac{2\cdot a^3-9\cdot a\cdot(b-p)+27\cdot(c-q)}{2\cdot\left(a^2-3\cdot(b-p)\right)^{\frac{3}{2}}}\right)}{3}+\frac{\pi}{6}\right)-a}{3} \text{ or}$$

$$x=\frac{-\left(2\cdot\sqrt{a^2-3\cdot(b-p)}\cdot\sin\left(\frac{\sin^{-1}\left(\frac{2\cdot a^3-9\cdot a\cdot(b-p)+27\cdot(c-q)}{2\cdot\left(a^2-3\cdot(b-p)\right)^{\frac{3}{2}}}\right)}{3}+\frac{\pi}{3}\right)+a\right)}{3} \text{ or}$$

$$x=\frac{2\cdot\sqrt{a^2-3\cdot(b-p)}\cdot\sin\left(\frac{\sin^{-1}\left(\frac{2\cdot a^3-9\cdot a\cdot(b-p)+27\cdot(c-q)}{2\cdot\left(a^2-3\cdot(b-p)\right)^{\frac{3}{2}}}\right)}{3}\right)-a}{3}$$

$v:=\text{zeros}(f(x)-g(x),x)$

$$▸\left\{\frac{2\cdot\sqrt{a^2-3\cdot(b-p)}\cdot\cos\left(\frac{\sin^{-1}\left(\frac{2\cdot a^3-9\cdot a\cdot(b-p)+27\cdot(c-q)}{2\cdot\left(a^2-3\cdot(b-p)\right)^{\frac{3}{2}}}\right)}{3}+\frac{\pi}{6}\right)-a}{3}, \frac{-\left(2\cdot\sqrt{a^2-3\cdot(b-p)}\cdot\sin\left(\frac{\sin^{-1}\left(\frac{2\cdot}{}\right.}{}\right.\right.}{} ▸$$

$\text{sum}(v)$ ▸ $-a$

- *With reasoning*
  The proof is easy – also by hand – following the next strategy.

  The solutions $x1, x2, x3$ of the equation $x^3 + ax^2 + bx + c = px + q$ are the zeros of the polynomial $h(x) = f(x) - g(x) = x^3 + ax^2 + (b-p)x + c - q$ (1).

  The statement above implies $h(x) = (x - x1)(x - x2)(x - x3)$ (2).

  The equation (1)=(2) completes the proof.

  Cubic Zeros

  $f(x):=x^3+a\cdot x^2+b\cdot x+c$

  $g(x):=p\cdot x+q$

  $h1(x):=f(x)-g(x) = x^3+a\cdot x^2+(b-p)\cdot x+c-q$

  $h2(x):=\text{expand}((x-x1)\cdot(x-x2)\cdot(x-x3)) = x^3+(-x1-x2-x3)\cdot x^2+(x1\cdot(x2+x3)+x2\cdot x3)\cdot x-x1\cdot x2\cdot x3$

  $h1(x)=h2(x)$ ▸ $x^3+a\cdot x^2+(b-p)\cdot x+c-q=x^3+(-x1-x2-x3)\cdot x^2+(x1\cdot(x2+x3)+x2\cdot x3)\cdot x-x1\cdot x2\cdot x3$

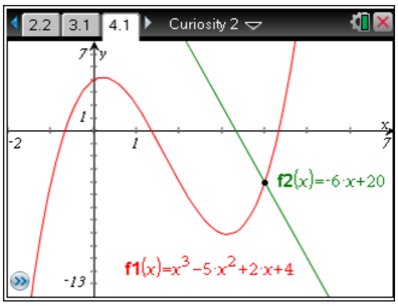
  Because *p* and *q* do not affect the coefficient of the 2nd degree term of the cubic function $f(x) - g(x)$ the sum of the intersection points for all lines is always the same.

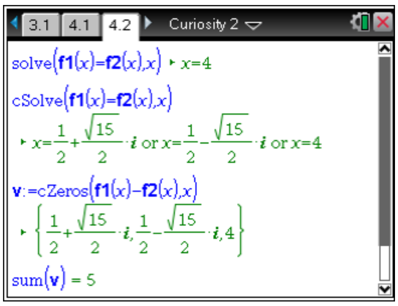
This example demonstrates the importance of good mathematical understanding for solving problems with CAS. Students will not be able to use CAS to solve a problem unless they have good mathematical understanding of the problem.

Three other situations

(i) $f(x) = x^3 - 5x^2 + 2x + 4$ and $g(x) = -6x + 20$

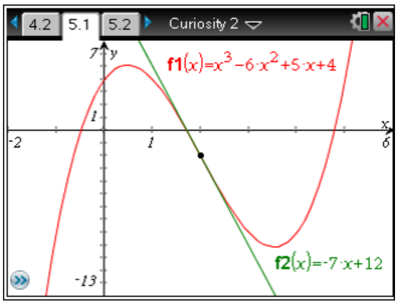
There is only one intersection point. The equation $f(x) = g(x)$ has one real solution and two complex solutions. The reasoning above can be easily extended to complex numbers.

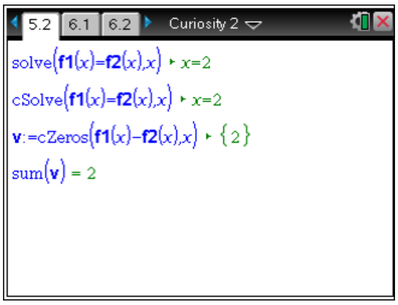




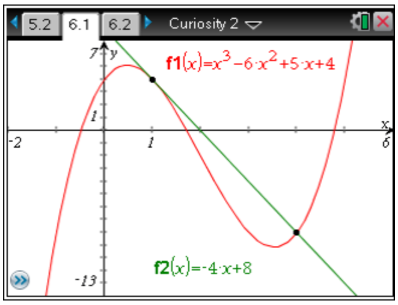
(ii) $f(x) = x^3 - 6x^2 + 5x + 4$ and $g(x) = -7x + 12$

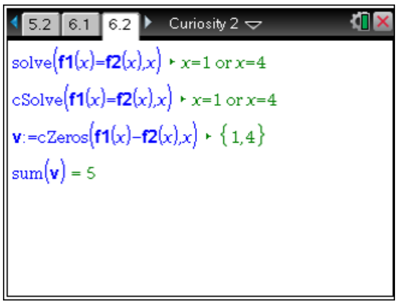
The sum of the zeros is not equal to 6! Why?





(iii) $f(x) = x^3 - 6x^2 + 5x + 4$ and $g(x) = -4x + 8$

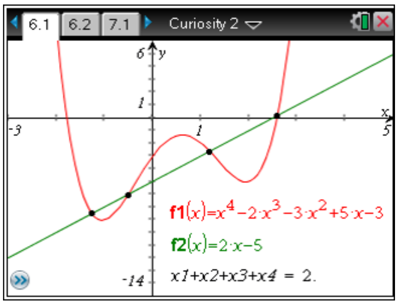


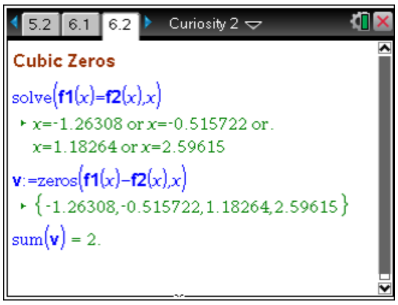


Conclusion?

Based on all the situations above, conjecture a general theorem about the sum of solutions for any equation $x^3 + ax^2 + bx + c = px + q$ with $a, b, c, p, q \in$ ° .

Can we generalization for 4th order polynomials?





And 5th, 6th, …., nth, …?

Curiosity 3 – Cubic Sum of Integers

Based on *Connecting Representations and Representing Connections*

A. M. Martínez Cruz - Northern Arizona University

Property

If a, b, c, d are consecutive integers then $a + b^2 + c^3$ is divisible by d.

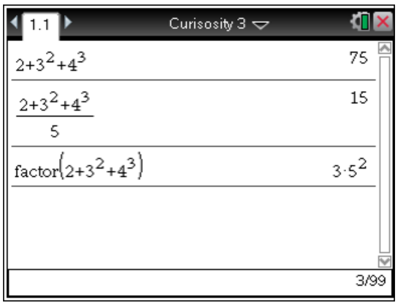
We will represent/investigate this statement numerically, graphically and algebraically.

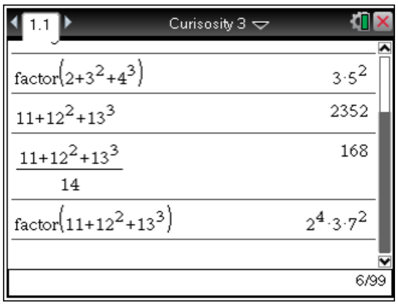
Numeric approach

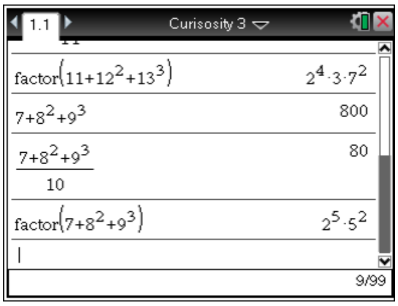
Take four consecutive integers - a, b, c, d - make the sum $a + b^2 + c^3$ and divide this sum by d .

Do this for several quadruplets of consecutive integers.

Factor the sum into prime factors and explain why the property is true.





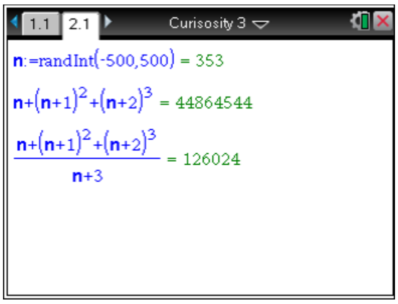


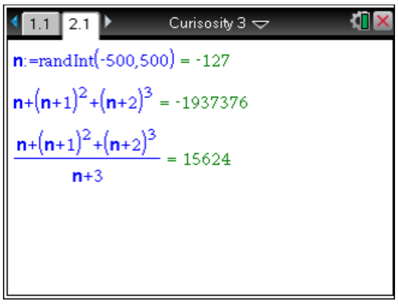
Symbolic generalization

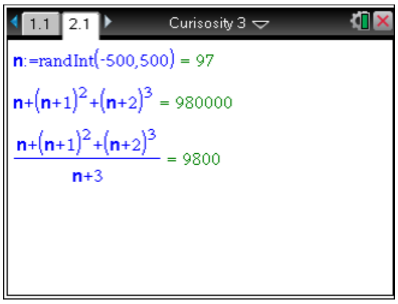
Define n as a random integer between -500 and 500: $\text{randInt}(-500, 500)$.

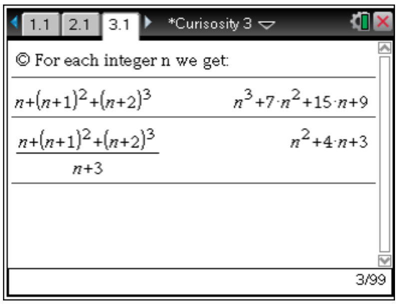
Write the ratio $\dfrac{a + b^2 + c^3}{d}$ of four consecutive numbers a, b, c, d in term of n being the first integer.

Factor for each n the sum into prime factors and explain why the property is true.







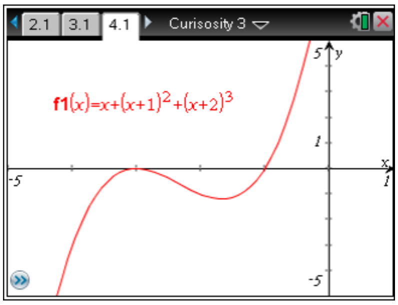


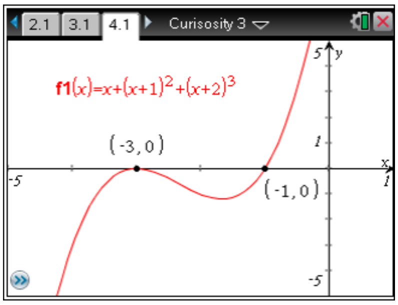
Graphical approach

Define the function $f1(x)$ as the sum of four consecutive integers with x being the first integer.

Plot the graph with the following Window Settings: $\begin{cases} \text{XMIN} = -5 \\ \text{XMAX} = 1 \\ \text{YMIN} = -5 \\ \text{YMAX} = 5 \end{cases}$ and $\text{XScale} = \text{YScale} = 1$.

What does the graph of $f1$ tells you about divisibility of the sum related to the four integers?





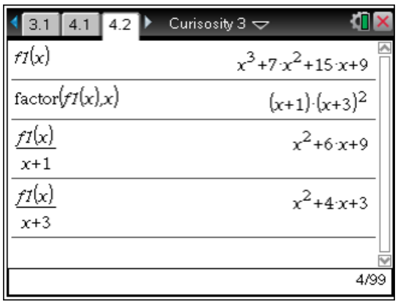
Symbolic Approach

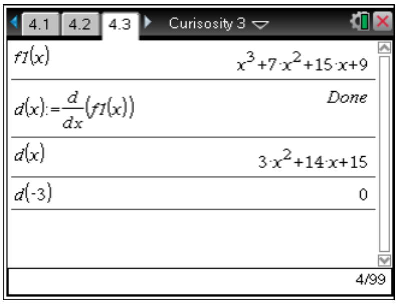
Use CAS to confirm what we have learned in the numeric and graphical approach.

The graph of $f1(x) = x + (x+1)^2 + (x+2)^3$ tells us $f1(-1) = f(-3) = 0$ which yields

$(x+1)$ and $(x+3)$ are factors of $f1(x)$.

And the first derivative shows us that $(x+3)$ is a double factor.



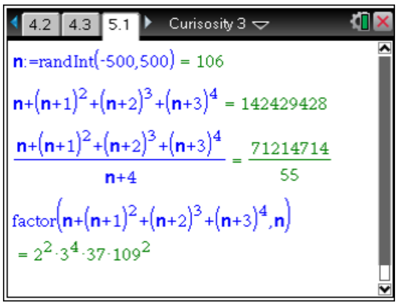


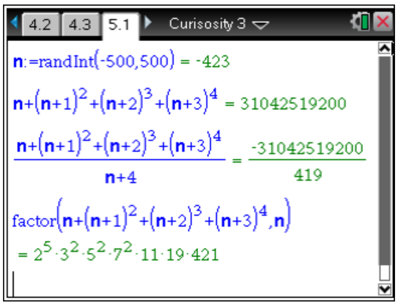
Generalization of the property

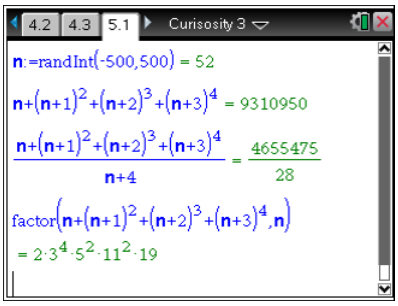
Let's check if the property can be extended to more than four consecutive integers.

Does e divide $a + b^2 + c^3 + d^4$ when a, b, c, d, e are five consecutive integers?

Quick calculations deliver counter examples for this statement.

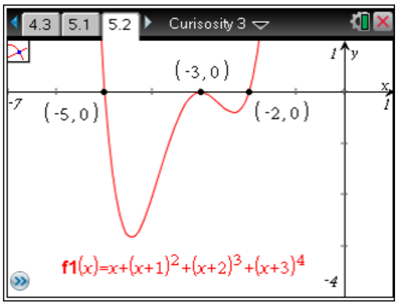


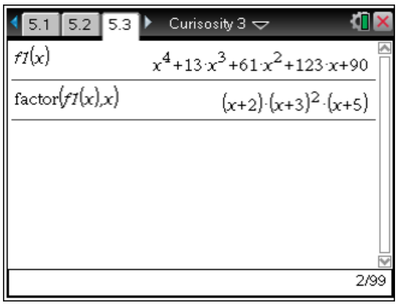


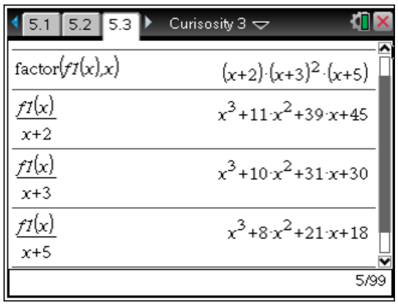


What does “plotting the graph of $f(x)=x+(x+1)^2+(x+2)^3+(x+3)^4$ “ or “factorizing $x+(x+1)^2+(x+2)^3+(x+3)^4$ “ tells you?

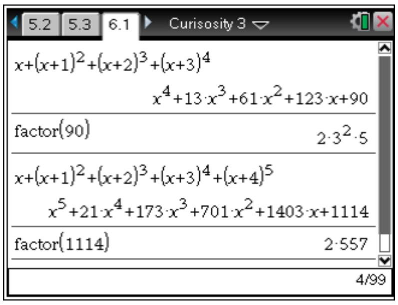
Which integers z divide $z+(z+1)^2+(z+2)^3+(z+3)^4$?







According to the rational roots theorem – *the possible rational number roots of a polynomial are given by the divisors of the independent term divided by the divisors of the leading coefficient* – the generalization for five consecutive integers cannot be true.



The constant term of $x+(x+1)^2+(x+2)^3+(x+3)^4$ is 90.

The prime factorization of $90=2\cdot 5\cdot 3^2$ shows that 4 is not a divisor of 90.

Same reasoning shows that the property cannot be extended to six consecutive integers.

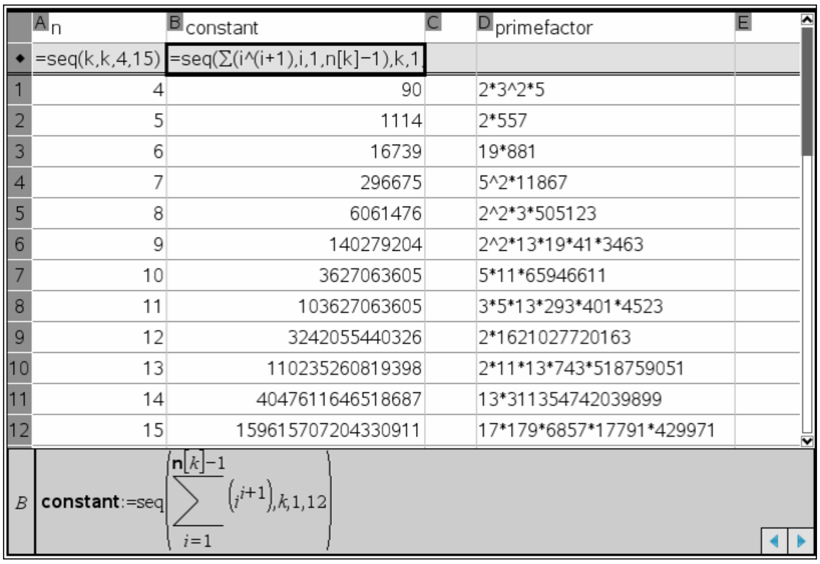
To see if we can extend the theorem to more than four consecutive integers we check the prime factorization of the constant term.

The constant term of $x+(x+1)^2+(x+2)^3+(x+3)^4$ is $1^2+2^3+3^4$,

$x+(x+1)^2+(x+2)^3+(x+3)^4+(x+4)^5$ is $1^2+2^3+3^4+4^5$

………

The spreadsheet below shows us that the property we started from cannot be extended up to 15.



	A n	B constant	C	D primefactor	E
◆	=seq(k,k,4,15)	=seq(Σ(i^(i+1),i,1,n[k]−1),k,1			
1	4	90		2*3^2*5	
2	5	1114		2*557	
3	6	16739		19*881	
4	7	296675		5^2*11867	
5	8	6061476		2^2*3*505123	
6	9	140279204		2^2*13*19*41*3463	
7	10	3627063605		5*11*65946611	
8	11	103627063605		3*5*13*293*401*4523	
9	12	3242055440326		2*1621027720163	
10	13	110235260819398		2*11*13*743*518759051	
11	14	4047611646518687		13*311354742039899	
12	15	159615707204330911		17*179*6857*17791*429971	

B constant:=seq$\left(\sum_{i=1}^{n[k]-1}\left(i^{i+1}\right),k,1,12\right)$

A teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

George Pólya - How to Solve it