International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Mathematics

Higher level

Specimen paper 3

For first examinations in 2014

CONTENTS

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 3 – DISCRETE MATHEMATICS

SPECIMEN

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 9]*

(a) Use the Euclidean algorithm to find the greatest common divisor of 259 and 581. *[4 marks]*

(b) Hence, or otherwise, find the general solution to the diophantine equation $259x + 581y = 7$. *[5 marks]*

2. *[Maximum mark: 13]*

The graph G has vertices P, Q, R, S, T and the following table shows the number of edges joining each pair of vertices.

	P	**Q**	**R**	**S**	**T**
P	0	1	0	1	2
Q	1	0	1	0	0
R	0	1	0	1	1
S	1	0	1	0	0
T	2	0	1	0	0

(a) Draw the graph G as a planar graph. *[2 marks]*

(b) Giving a reason, state whether or not G is

(i) simple;

(ii) connected;

(iii) bipartite. *[4 marks]*

(c) Explain what feature of G enables you to state that it has an Eulerian trail and write down a trail. *[2 marks]*

(This question continues on the following page)

(Question 2 continued)

(d) Explain what feature of G enables you to state it does not have an Eulerian circuit. *[1 mark]*

(e) Find the maximum number of edges that can be added to the graph G (not including any loops or further multiple edges) whilst still keeping it planar. *[4 marks]*

3. *[Maximum mark: 12]*

(a) One version of Fermat's little theorem states that, under certain conditions, $a^{p-1} \equiv 1 \pmod p$.

(i) Show that this result is not true when $a = 2$, $p = 9$ and state which of the conditions is not satisfied.

(ii) Find the smallest positive value of k satisfying the congruence $2^{45} \equiv k \pmod 9$. *[6 marks]*

(b) Find all the integers between 100 and 200 satisfying the simultaneous congruences $3x \equiv 4 \pmod 5$ and $5x \equiv 6 \pmod 7$. *[6 marks]*

4. *[Maximum mark: 12]*

The weights of the edges of a graph G with vertices A, B, C, D and E are given in the following table.

	A	**B**	**C**	**D**	**E**
A	–	11	18	12	9
B	11	–	17	13	14
C	18	17	–	16	10
D	12	13	16	–	15
E	9	14	10	15	–

(a) Starting at A, use the nearest neighbour algorithm to find an upper bound for the travelling salesman problem for G. *[4 marks]*

(b) (i) Use Kruskal's algorithm to find and draw a minimum spanning tree for the subgraph obtained by removing the vertex A from G.

(ii) Hence use the deleted vertex algorithm to find a lower bound for the travelling salesman problem for G. *[8 marks]*

5. *[Maximum mark: 14]*

(a) The sequence $\{u_n\}$, $n \in \mathbb{Z}^+$, satisfies the recurrence relation $u_{n+2} = 5u_{n+1} - 6u_n$. Given that $u_1 = u_2 = 3$, obtain an expression for u_n in terms of n. *[6 marks]*

(b) The sequence $\{v_n\}$, $n \in \mathbb{Z}^+$, satisfies the recurrence relation $v_{n+2} = 4v_{n+1} - 4v_n$. Given that $v_1 = 2$ and $v_2 = 12$, use the principle of strong mathematical induction to show that $v_n = 2^n (2n-1)$. *[8 marks]*

SPEC/5/MATHL/HP3/ENG/TZ0/DM/M

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MARKSCHEME

SPECIMEN

MATHEMATICS
DISCRETE MATHEMATICS

Higher Level

Paper 3

9 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*., do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty **(AP)** no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

*The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award **A0**. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $581 = 2 \times 259 + 63$ ***M1A1***

$259 = 4 \times 63 + 7$ ***A1***

$63 = 9 \times 7$

the GCD is therefore 7 ***A1***

[4 marks]

(b) consider

$7 = 259 - 4 \times 63$ ***M1***

$= 259 - 4 \times (581 - 2 \times 259)$ ***A1***

$= 259 \times 9 + 581 \times (-4)$ ***A1***

the general solution is therefore

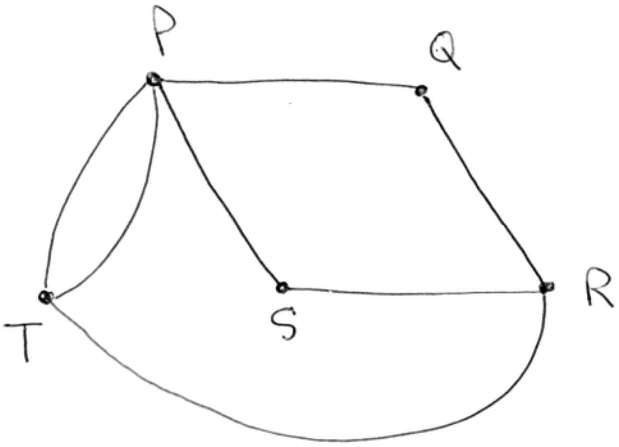
$x = 9 + 83n$; $y = -4 - 37n$ where $n \in \mathbb{Z}$ ***M1A1***

Notes:	Accept solutions laid out in tabular form. Dividing the diophantine equation by 7 is an equally valid method.

[5 marks]

Total [9 marks]

2. (a)



A2

[2 marks]

(b) (i) *G* is not simple because 2 edges join P to T **R1**

(ii) *G* is connected because there is a path joining every pair of vertices **R1**

(iii) (P, R) and $(\mathrm{Q}, \mathrm{S}, \mathrm{T})$ are disjoint vertices **R1**
so *G* is bipartite **A1**

Note: Award the A1 only if the R1 is awarded.

[4 marks]

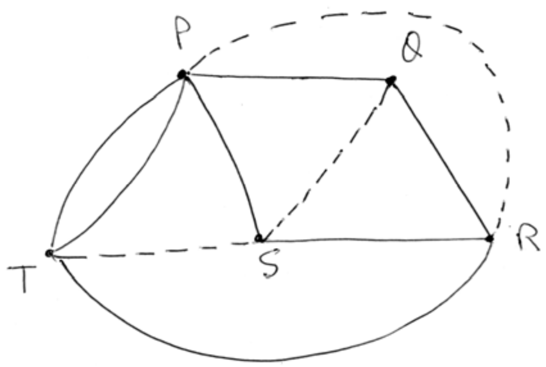
(c) *G* has an Eulerian trail because it has two vertices of odd degree (R and T have degree 3), all the other vertices having even degree **R1**
the following example is such a trail
TPTRSPQR **A1**

[2 marks]

(d) *G* has no Eulerian circuit because there are 2 vertices which have odd degree **R1**

[1 mark]

(e) consider



so it is possible to add 3 extra edges **A1**
consider *G* with one of the edges PT deleted; this is a simple graph with 6 edges; on addition of the new edges, it will still be simple **M1**
$e \le 3v - 6 \Rightarrow e \le 3 \times 5 - 6 = 9$ **R1**
so at most 3 edges can be added **R1**

[4 marks]

Total [13 marks]

3. (a) (i) $2^8 = 256 \equiv 4 \pmod 9$ (so not true) ***A1***

9 is not prime ***A1***

(ii) consider various powers of 2, *e.g.* obtaining ***M1***

$2^6 = 64 \equiv 1 \pmod 9$ ***A1***

therefore

$2^{45} = (2^6)^7 \times 2^3$ ***M1***

$\equiv 8 \pmod 9$ (so $k = 8$) ***A1***

[6 marks]

(b) **EITHER**

the solutions to $3x \equiv 4 \pmod 5$ are 3, 8, 13, 18, 23,… ***M1A1***

the solutions to $5x \equiv 6 \pmod 7$ are 4, 11, 18,… ***A1***

18 is therefore the smallest solution ***A1***

the general solution is

$18 + 35n,\ n \in \mathbb{Z}$ ***M1***

the required solutions are therefore 123, 158, 193 ***A1***

OR

$3x \equiv 4 \pmod 5 \Rightarrow 2 \times 3x \equiv 2 \times 4 \pmod 5 \Rightarrow x \equiv 3 \pmod 5$ ***A1***

$\Rightarrow x = 3 + 5t$ ***M1***

$\Rightarrow 15 + 25t \equiv 6 \pmod 7 \Rightarrow 4t \equiv 5 \pmod 7 \Rightarrow 2 \times 4t \equiv 2 \times 5 \pmod 7 \Rightarrow t \equiv 3 \pmod 7$ ***A1***

$\Rightarrow t = 3 + 7n$ ***A1***

$\Rightarrow x = 3 + 5(3 + 7n) = 18 + 35n$ ***M1***

the required solutions are therefore 123, 158, 193 ***A1***

OR

using the Chinese remainder theorem formula method

first convert the congruences to $x \equiv 3 \pmod 5$ and $x \equiv 4 \pmod 7$ ***A1A1***

$M = 35,\ M_1 = 7,\ M_2 = 5,\ m_1 = 5,\ m_2 = 7,\ a_1 = 3,\ a_2 = 4$

x_1 is the solution of $M_1 x_1 \equiv 1 \pmod{m_1}$, *i.e.* $7x_1 \equiv 1 \pmod 5$ so $x_1 = 3$

x_2 is the solution of $M_2 x_2 \equiv 1 \pmod{m_2}$, *i.e.* $5x_2 \equiv 1 \pmod 7$ so $x_2 = 3$

a solution is therefore

$x = a_1 M_1 x_1 + a_2 M_2 x_2$ ***M1***

$= 3 \times 7 \times 3 + 4 \times 5 \times 3 = 123$ ***A1***

the general solution is $123 + 35n,\ n \in \mathbb{Z}$ ***M1***

the required solutions are therefore 123, 158, 193 ***A1***

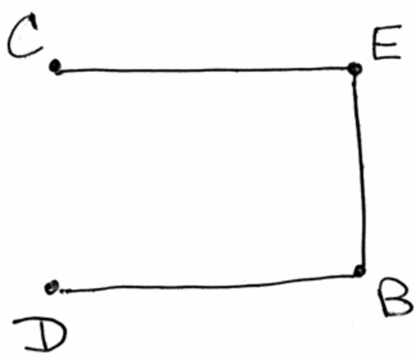
[6 marks]

Total [12 marks]

4. (a) using the nearest neighbour algorithm, starting with A,
$A \rightarrow E$, $E \rightarrow C$ ***A1***
$C \rightarrow D$, $D \rightarrow B$ ***A1***
$B \rightarrow A$ ***A1***
the upper bound is therefore $9+10+16+13+11=59$ ***A1***

[4 marks]

(b) (i) the edges are added in the order CE ***A1***
BD ***A1***
BE ***A1***



A1

(ii) the weight of the minimum spanning tree is 37 ***(A1)***
we now reconnect A with the 2 edges of least weight ***(M1)***
i.e. AE and AB ***A1***
the lower bound is therefore $37+9+11=57$ ***A1***

[8 marks]

Total [12 marks]

5. (a) the auxiliary equation is

$m^2 - 5m + 6 = 0$ ***M1***

giving $m = 2, 3$ ***A1***

the general solution is

$u_n = A \times 2^n + B \times 3^n$ ***A1***

substituting $n = 1, 2$

$2A + 3B = 3$ ***M1***

$4A + 9B = 3$ ***A1***

the solution is $A = 3,\ B = -1$ giving $u_n = 3 \times 2^n - 3^n$ ***A1***

[6 marks]

(b) we first prove that $v_n = 2^n(2n-1)$ for $n = 1, 2$ ***M1***

for $n = 1$, it gives $2 \times 1 = 2$ which is correct

for $n = 2$, it gives $4 \times 3 = 12$ which is correct ***A1***

we now assume that the result is true for $n \le k$ ***M1***

consider

$v_{k+1} = 4v_k - 4v_{k-1}\ \ (k \ge 2)$ ***M1***

$= 4.2^k(2k-1) - 4.2^{k-1}(2k-3)$ ***A1***

$= 2^{k+1}(4k - 2 - 2k + 3)$ ***A1***

$= 2^{k+1}\left(2(k+1) - 1\right)$ ***A1***

this proves that if the result is true for $n \le k$ then it is true for $n \le k+1$

since we have also proved it true for $n \le 2$, the general result is proved by induction ***R1***

Note:	A reasonable attempt has to be made to the induction step for the final ***R1*** to be awarded.

[8 marks]

Total [14 marks]

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 3 – CALCULUS

SPECIMEN

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

3 pages

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 14]*

The function f is defined on the domain $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ by $f(x) = \ln(1+\sin x)$.

(a) Show that $f''(x) = -\dfrac{1}{(1+\sin x)}$. *[4 marks]*

(b) (i) Find the Maclaurin series for $f(x)$ up to and including the term in x^4.

(ii) Explain briefly why your result shows that f is neither an even function nor an odd function. *[7 marks]*

(c) Determine the value of $\lim\limits_{x \to 0} \dfrac{\ln(1+\sin x) - x}{x^2}$. *[3 marks]*

2. *[Maximum mark: 8]*

Consider the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y + \sqrt{x^2 - y^2},\ x > 0,\ x^2 > y^2.$$

(a) Show that this is a homogeneous differential equation. *[1 mark]*

(b) Find the general solution, giving your answer in the form $y = f(x)$. *[7 marks]*

3. *[Maximum mark: 15]*

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^x + y\tan x \text{, given that } y = 1 \text{ when } x = 0 .$$

The domain of the function y is $\left[0, \frac{\pi}{2}\right[$.

(a) By finding the values of successive derivatives when $x = 0$, find the Maclaurin series for y as far as the term in x^3. *[6 marks]*

(b) (i) Differentiate the function $\mathrm{e}^x(\sin x + \cos x)$ and hence show that

$$\int \mathrm{e}^x \cos x \,\mathrm{d}x = \frac{1}{2}\mathrm{e}^x(\sin x + \cos x) + c .$$

(ii) Find an integrating factor for the differential equation and hence find the solution in the form $y = f(x)$. *[9 marks]*

4. *[Maximum mark: 10]*

Let $f(x) = 2x + |x|,\ x \in \mathbb{R}$.

(a) Prove that f is continuous but not differentiable at the point $(0, 0)$. *[7 marks]*

(b) Determine the value of $\int_{-a}^{a} f(x)\,\mathrm{d}x$ where $a > 0$. *[3 marks]*

5. *[Maximum mark: 13]*

Consider the infinite series $\sum_{n=1}^{\infty} \frac{(n-1)\,x^n}{n^2 \times 2^n}$.

(a) Find the radius of convergence. *[4 marks]*

(b) Find the interval of convergence. *[9 marks]*

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MARKSCHEME

SPECIMEN

MATHEMATICS
CALCULUS

Higher Level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*., do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty **(AP)** no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

*The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award **A0**. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation

The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $f'(x) = \frac{\cos x}{1 + \sin x}$ ***A1***

$f''(x) = \frac{-\sin x(1 + \sin x) - \cos x \cos x}{(1 + \sin x)^2}$ ***M1A1***

$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2}$ ***A1***

$= -\frac{1}{1 + \sin x}$ ***AG***

[4 marks]

(b) (i) $f'''(x) = \frac{\cos x}{(1 + \sin x)^2}$ ***A1***

$f^{(4)}(x) = \frac{-\sin x(1 + \sin x)^2 - 2(1 + \sin x)\cos^2 x}{(1 + \sin x)^4}$ ***M1A1***

$f(0) = 0,\ f'(0) = 1,\ f''(0) = -1$ ***M1***

$f'''(0) = 1,\ f^{(4)}(0) = -2$ ***A1***

$f(x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$ ***A1***

(ii) the series contains even and odd powers of x ***R1***

[7 marks]

(c) $\lim_{x \to 0} \frac{\ln(1 + \sin x) - x}{x^2} = \lim_{x \to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{6} + \ldots - x}{x^2}$ ***M1***

$= \lim_{x \to 0} \frac{\frac{-1}{2} + \frac{x}{6} + \ldots}{1}$ ***(A1)***

$= -\frac{1}{2}$ ***A1***

Note: Use of l'Hopital's Rule is also acceptable.

[3 marks]

Total [14 marks]

2. (a) the equation can be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y + \sqrt{x^2 - y^2}}{x} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}$$ ***A1***

so the differential equation is homogeneous ***AG***

[1 mark]

(b) put $y = vx$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1A1***

substituting,

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + \sqrt{1 - v^2}$$ ***M1***

$$\int \frac{\mathrm{d}v}{\sqrt{1 - v^2}} = \int \frac{\mathrm{d}x}{x}$$ ***M1***

$$\arcsin v = \ln x + C$$ ***A1***

$$\frac{y}{x} = \sin(\ln x + C)$$ ***A1***

$$y = x \sin(\ln x + C)$$ ***A1***

[7 marks]

Total [8 marks]

3. (a) we note that $y(0)=1$ and $y'(0)=2$ ***A1***

$y''=2\mathrm{e}^x+y'\tan x+y\sec^2 x$ ***M1***

$y''(0)=3$ ***A1***

$y'''=2\mathrm{e}^x+y''\tan x+2y'\sec^2 x+2y\sec^2 x\tan x$ ***M1***

$y'''(0)=6$ ***A1***

the maclaurin series solution is therefore

$y=1+2x+\dfrac{3x^2}{2}+x^3+\dots$ ***A1***

[6 marks]

(b) (i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x(\sin x+\cos x)\right)=\mathrm{e}^x(\sin x+\cos x)+\mathrm{e}^x(\cos x-\sin x)$ ***M1***

$=2\mathrm{e}^x\cos x$ ***A1***

it follows that

$\int \mathrm{e}^x\cos x\,\mathrm{d}x=\dfrac{1}{2}\mathrm{e}^x(\sin x+\cos x)+c$ ***AG***

(ii) the differential equation can be written as

$\dfrac{\mathrm{d}y}{\mathrm{d}x}-y\tan x=2\mathrm{e}^x$ ***M1***

$\mathrm{IF}=\mathrm{e}^{\int -\tan x\,\mathrm{d}x}=\mathrm{e}^{\ln\cos x}=\cos x$ ***M1A1***

$\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x}-y\sin x=2\mathrm{e}^x\cos x$ ***M1***

integrating,

$y\cos x=\mathrm{e}^x(\sin x+\cos x)+C$ ***A1***

$y=1$ when $x=0$ gives $C=0$ ***M1***

therefore

$y=\mathrm{e}^x(1+\tan x)$ ***A1***

[9 marks]

Total [15 marks]

4. (a) we note that $f(0)=0$, $f(x)=3x$ for $x>0$ and $f(x)=x$ for $x<0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0$ ***M1A1***

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x = 0$ ***A1***

since $f(0)=0$, the function is continuous when $x=0$ ***AG***

$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h}{h} = 1$ ***M1A1***

$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{3h}{h} = 3$ ***A1***

these limits are unequal ***R1***

so f is not differentiable when $x=0$ ***AG***

[7 marks]

(b) $\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} x\,\mathrm{d}x + \int_{0}^{a} 3x\,\mathrm{d}x$ ***M1***

$= \left[\frac{x^2}{2}\right]_{-a}^{0} + \left[\frac{3x^2}{2}\right]_{0}^{a}$ ***A1***

$= a^2$ ***A1***

[3 marks]

Total [10 marks]

5. (a) using the ratio test, $\dfrac{u_{n+1}}{u_n} = \dfrac{nx^{n+1}}{(n+1)^2\, 2^{n+1}} \times \dfrac{n^2\, 2^n}{(n-1)\, x^n}$ ***M1***

$$= \frac{n^3}{(n+1)^2 (n-1)} \times \frac{x}{2}$$ ***A1***

$\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \dfrac{x}{2}$ ***A1***

the radius of convergence R satisfies

$\dfrac{R}{2} = 1$ so $R = 2$ ***A1***

[4 marks]

(b) considering $x = 2$ for which the series is

$\sum\limits_{n=1}^{\infty} \dfrac{(n-1)}{n^2}$

using the limit comparison test with the harmonic series ***M1***

$\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, which diverges

consider

$\lim\limits_{n\to\infty} \dfrac{u_n}{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{n-1}{n} = 1$ ***A1***

the series is therefore divergent for $x = 2$ ***A1***

when $x = -2$, the series is

$\sum\limits_{n=1}^{\infty} \dfrac{(n-1)}{n^2} \times (-1)^n$

this is an alternating series in which the n^{th} term tends to 0 as $n \to \infty$ ***A1***

consider $f(x) = \dfrac{x-1}{x^2}$ ***M1***

$f'(x) = \dfrac{2-x}{x^3}$ ***A1***

this is negative for $x > 2$ so the sequence $\{|u_n|\}$ is eventually decreasing ***R1***

the series therefore converges when $x = -2$ by the alternating series test ***R1***

the interval of convergence is therefore $[-2, 2[$ ***A1***

[9 marks]

Total [13 marks]

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 3 – SETS, RELATIONS AND GROUPS

SPECIMEN

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

3 pages

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 14]*

(a) The relation R is defined on $\mathbb{Z}^+$ by aRb if and only if ab is even. Show that only one of the conditions for R to be an equivalence relation is satisfied. *[5 marks]*

(b) The relation S is defined on $\mathbb{Z}^+$ by aSb if and only if $a^2 \equiv b^2 (\text{mod } 6)$.

(i) Show that S is an equivalence relation.

(ii) For each equivalence class, give the four smallest members. *[9 marks]*

2. *[Maximum mark: 13]*

The binary operations $\odot$ and $*$ are defined on $\mathbb{R}^+$ by

$$a \odot b = \sqrt{ab} \text{ and } a * b = a^2 b^2 .$$

Determine whether or not

(a) $\odot$ is commutative; *[2 marks]*

(b) $*$ is associative; *[4 marks]*

(c) $*$ is distributive over $\odot$; *[4 marks]*

(d) $\odot$ has an identity element. *[3 marks]*

3. *[Maximum mark: 16]*

The group $\{G, \times_7\}$ is defined on the set $\{1, 2, 3, 4, 5, 6\}$ where $\times_7$ denotes multiplication modulo 7.

(a) (i) Write down the Cayley table for $\{G, \times_7\}$.

(ii) Determine whether or not $\{G, \times_7\}$ is cyclic.

(iii) Find the subgroup of G of order 3, denoting it by H.

(iv) Identify the element of order 2 in G and find its coset with respect to H. *[10 marks]*

(b) The group $\{K, \circ\}$ is defined on the six permutations of the integers 1, 2, 3 and $\circ$ denotes composition of permutations.

(i) Show that $\{K, \circ\}$ is non-Abelian.

(ii) Giving a reason, state whether or not $\{G, \times_7\}$ and $\{K, \circ\}$ are isomorphic. *[6 marks]*

4. *[Maximum mark: 9]*

The groups $\{G, *\}$ and $\{H, \odot\}$ are defined by the following Cayley tables.

G

$*$	$\boldsymbol{E}$	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C}$
$\boldsymbol{E}$	E	A	B	C
$\boldsymbol{A}$	A	E	C	B
$\boldsymbol{B}$	B	C	A	E
$\boldsymbol{C}$	C	B	E	A

H

$\odot$	$\boldsymbol{e}$	$\boldsymbol{a}$
$\boldsymbol{e}$	e	a
$\boldsymbol{a}$	a	e

By considering a suitable function from G to H, show that a surjective homomorphism exists between these two groups. State the kernel of this homomorphism.

5. *[Maximum mark: 8]*

Let $\{G, *\}$ be a finite group and let H be a non-empty subset of G. Prove that $\{H, *\}$ is a group if H is closed under $*$.

SPEC/5/MATHL/HP3/ENG/TZ0/SG/M

MARKSCHEME

SPECIMEN

MATHEMATICS
SETS, RELATIONS AND GROUPS

Higher Level

Paper 3

7 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g*. ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*., do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty ***(AP)*** *no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award ***A0****. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) reflexive: if a is odd, $a \times a$ is odd so R is not reflexive ***R1***
symmetric: if ab is even then ba is even so R is symmetric ***R1***
transitive: let aRb and bRc; it is necessary to determine whether or not aRc ***(M1)***
for example $5R2$ and $2R3$ ***A1***
since 5×3 is not even, 5 is not related to 3 and R is not transitive ***R1***

[5 marks]

(b) (i) reflexive: $a^2 \equiv a^2 (\mathrm{mod}\ 6)$ so S is reflexive ***R1***
symmetric: $a^2 \equiv b^2(\mathrm{mod}\,6) \Rightarrow 6\,|\,(a^2 - b^2) \Rightarrow 6\,|\,(b^2 - a^2) \Rightarrow b^2 \equiv a^2(\mathrm{mod}\,6)$ ***R1***
so S is symmetric
transitive: let aSb and bSc so that $a^2 = b^2 + 6M$ and $b^2 = c^2 + 6N$ ***M1***
it follows that $a^2 = c^2 + 6(M + N)$ so aSc and S is transitive ***R1***
S is an equivalence relation because it satisfies the three conditions ***AG***

(ii) by considering the squares of integers (mod 6), the equivalence classes are ***(M1)***
$\{1, 5, 7, 11, \ldots\}$ ***A1***
$\{2, 4, 8, 10, \ldots\}$ ***A1***
$\{3, 9, 15, 21, \ldots\}$ ***A1***
$\{6, 12, 18, 24, \ldots\}$ ***A1***

[9 marks]

Total [14 marks]

2. (a) $a \odot b = \sqrt{ab} = \sqrt{ba} = b \odot a$ ***A1***
since $a \odot b = b \odot a$ it follows that $\odot$ is commutative ***R1***

[2 marks]

(b) $a * (b * c) = a * b^2c^2 = a^2b^4c^4$ ***M1A1***
$(a * b) * c = a^2b^2 * c = a^4b^4c^2$ ***A1***
these are different, therefore $*$ is not associative ***R1***

Note: Accept numerical counter-example.

[4 marks]

(c) $a * (b \odot c) = a * \sqrt{bc} = a^2bc$ ***M1A1***
$(a * b) \odot (a * c) = a^2b^2 \odot a^2c^2 = a^2bc$ ***A1***
these are equal so $*$ is distributive over $\odot$ ***R1***

[4 marks]

(d) the identity e would have to satisfy
$a \odot e = a$ for all a ***M1***
now $a \odot e = \sqrt{ae} = a \Rightarrow e = a$ ***A1***
therefore there is no identity element ***A1***

[3 marks]

Total [13 marks]

3. (a) (i) the Cayley table is

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

A3

Note: Deduct 1 mark for each error up to a maximum of 3.

(ii) by considering powers of elements, ***(M1)***
it follows that 3 (or 5) is of order 6 ***A1***
so the group is cyclic ***A1***

(iii) we see that 2 and 4 are of order 3 so the subgroup of order 3 is $\{1, 2, 4\}$ ***M1A1***

(iv) the element of order 2 is 6 ***A1***
the coset is $\{3, 5, 6\}$ ***A1***

[10 marks]

(b) (i) consider for example

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$ ***M1A1***

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$ ***M1A1***

Note: Award ***M1A1M1A0*** if both compositions are done in the wrong order.

Note: Award ***M1A1M0A0*** if the two compositions give the same result, if no further attempt is made to find two permutations which are not commutative.

these are different so the group is not Abelian ***R1AG***

(ii) they are not isomorphic because $\{G, \times_7\}$ is Abelian and $\{K, \circ\}$ is not ***R1***

[6 marks]

Total [16 marks]

4. consider the function f given by

$$f(E)=e$$
$$f(A)=e$$
$$f(B)=a \qquad \textbf{\textit{M1A1}}$$
$$f(C)=a$$

then, it has to be shown that

$$f(X*Y)=f(X)\odot f(Y) \text{ for all } X, Y\in G \qquad \textbf{\textit{(M1)}}$$

consider

$f\big((E \text{ or } A)*(E \text{ or } A)\big)=f(E \text{ or } A)=e;\ f(E \text{ or } A)\odot f(E \text{ or } A)=e\odot e=e$ ***M1A1***

$f\big((E \text{ or } A)*(B \text{ or } C)\big)=f(B \text{ or } C)=a;\ f(E \text{ or } A)\odot f(B \text{ or } C)=e\odot a=a$ ***A1***

$f\big((B \text{ or } C)*(B \text{ or } C)\big)=f(E \text{ or } A)=e;\ f(B \text{ or } C)\odot f(B \text{ or } C)=a\odot a=e$ ***A1***

since the groups are Abelian, there is no need to consider $f\big((B \text{ or } C)*(E \text{ or } A)\big)$ ***R1***

the required property is satisfied in all cases so the homomorphism exists

Note: A comprehensive proof using tables is acceptable.

the kernel is $\{E, A\}$ ***A1***

[9 marks]

5. the associativity property carries over from G ***R1***

closure is given ***R1***

let $h\in H$ and let n denote the order of h, (this is finite because G is finite) ***M1***

it follows that $h^n=e$, the identity element ***R1***

and since H is closed, $e\in H$ ***R1***

since $h*h^{n-1}=e$ ***M1***

it follows that h^{n-1} is the inverse, h^{-1}, of h ***R1***

and since H is closed, $h^{-1}\in H$ so each element of H has an inverse element ***R1***

the four requirements for H to be a group are therefore satisfied ***AG***

[8 marks]

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 3 – STATISTICS AND PROBABILITY

SPECIMEN

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

5 pages

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 10]*

A shopper buys 12 apples from a market stall and weighs them with the following results (in grams).

117, 124, 129, 118, 124, 116, 121, 126, 118, 121, 122, 129

You may assume that this is a random sample from a normal distribution with mean μ and variance σ^2.

(a) Determine unbiased estimates of μ and σ^2. *[3 marks]*

(b) Determine a 99 % confidence interval for μ. *[2 marks]*

(c) The stallholder claims that the mean weight of apples is 125 grams but the shopper claims that the mean is less than this.

(i) State suitable hypotheses for testing these claims.

(ii) Calculate the p-value of the above sample.

(iii) Giving a reason, state which claim is supported by your p-value using a 5 % significance level. *[5 marks]*

2. *[Maximum mark: 12]*

When Andrew throws a dart at a target, the probability that he hits it is $\frac{1}{3}$; when Bill throws a dart at the target, the probability that he hits the it is $\frac{1}{4}$. Successive throws are independent. One evening, they throw darts at the target alternately, starting with Andrew, and stopping as soon as one of their darts hits the target. Let X denote the total number of darts thrown.

(a) Write down the value of $\mathrm{P}(X=1)$ and show that $\mathrm{P}(X=2)=\frac{1}{6}$. *[2 marks]*

(b) Show that the probability generating function for X is given by

$$G(t)=\frac{2t+t^2}{6-3t^2}.$$ *[6 marks]*

(c) Hence determine $\mathrm{E}(X)$. *[4 marks]*

3. *[Maximum mark: 9]*

The weights of adult monkeys of a certain species are known to be normally distributed, the males with mean 30 kg and standard deviation 3 kg and the females with mean 20 kg and standard deviation 2.5 kg.

(a) Find the probability that the weight of a randomly selected male is more than twice the weight of a randomly selected female. *[5 marks]*

(b) Two males and five females stand together on a weighing machine. Find the probability that their total weight is less than 175 kg. *[4 marks]*

4. *[Maximum mark: 15]*

The students in a class take an examination in Applied Mathematics which consists of two papers. Paper 1 is in Mechanics and Paper 2 is in Statistics. The marks obtained by the students in Paper 1 and Paper 2 are denoted by (x, y) respectively and you may assume that the values of (x, y) form a random sample from a bivariate normal distribution with correlation coefficient ρ. The teacher wishes to determine whether or not there is a positive association between marks in Mechanics and marks in Statistics.

(a) State suitable hypotheses. *[1 mark]*

The marks obtained by the 12 students who sat both papers are given in the following table.

Student	A	B	C	D	E	F	G	H	I	J	K	L
x	52	47	82	69	38	50	72	46	23	60	42	53
y	55	44	79	62	41	37	71	44	31	45	47	49

(b) (i) Determine the product moment correlation coefficient for these data and state its p-value.

(ii) Interpret your p-value in the context of the problem. *[5 marks]*

(c) George obtained a mark of 63 on Paper 1 but was unable to sit Paper 2 because of illness. Predict the mark that he would have obtained on Paper 2. *[4 marks]*

(d) Another class of 16 students sat examinations in Physics and Chemistry and the product moment correlation coefficient between the marks in these two subjects was calculated to be 0.524. Using a 1 % significance level, determine whether or not this value suggests a positive association between marks in Physics and marks in Chemistry. *[5 marks]*

5. *[Maximum mark: 14]*

The discrete random variable X has the following probability distribution, where $0 < \theta < \frac{1}{3}$.

x	1	2	3
$\mathrm{P}(X = x)$	θ	2θ	$1 - 3\theta$

(a) Determine $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = 6\theta - 16\theta^2$. *[4 marks]*

In order to estimate θ, a random sample of n observations is obtained from the distribution of X.

(b) (i) Given that $\overline{X}$ denotes the mean of this sample, show that

$$\hat{\theta}_1 = \frac{3 - \overline{X}}{4}$$

is an unbiased estimator for θ and write down an expression for the variance of $\hat{\theta}_1$ in terms of n and θ.

(ii) Let Y denote the number of observations that are equal to 1 in the sample. Show that Y has the binomial distribution $\mathrm{B}(n, \theta)$ and deduce that $\hat{\theta}_2 = \frac{Y}{n}$ is another unbiased estimator for θ. Obtain an expression for the variance of $\hat{\theta}_2$.

(iii) Show that $\mathrm{Var}(\hat{\theta}_1) < \mathrm{Var}(\hat{\theta}_2)$ and state, with a reason, which is the more efficient estimator, $\hat{\theta}_1$ or $\hat{\theta}_2$. *[10 marks]*

MARKSCHEME

SPECIMEN

MATHEMATICS
STATISTICS AND PROBABILITY

Higher Level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*., do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

The method of dealing with accuracy errors on a whole paper basis by means of the Accuracy Penalty ***(AP)*** *no longer applies.*

Instructions to examiners about such numerical issues will be provided on a question by question basis within the framework of mathematical correctness, numerical understanding and contextual appropriateness.

The rubric on the front page of each question paper is given for the guidance of candidates. The markscheme (MS) may contain instructions to examiners in the form of "Accept answers which round to n significant figures (sf)". Where candidates state answers, required by the question, to fewer than n sf, award ***A0****. Some intermediate numerical answers may be required by the MS but not by the question. In these cases only award the mark(s) if the candidate states the answer exactly or to at least 2sf.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) unbiased estimate of $\mu = 122$ ***A1***

unbiased estimate of $\sigma^2 = 4.4406\ldots^2 = 19.7$ ***(M1)A1***

Note: Award ***(M1)A0*** for 4.44.

[3 marks]

(b) the 99 % confidence interval for μ is $[118, 126]$ ***A1A1***

[2 marks]

(c) (i) $\mathrm{H}_0 : \mu = 125$; $\mathrm{H}_1 : \mu < 125$ ***A1***

(ii) p-value $= 0.0220$ ***A2***

(iii) the shopper's claim is supported because $0.0220 < 0.05$ ***A1R1***

[5 marks]

Total [10 marks]

2. (a) $\mathrm{P}(X = 1) = \dfrac{1}{3}$ ***A1***

$\mathrm{P}(X = 2) = \dfrac{2}{3} \times \dfrac{1}{4}$ ***A1***

$= \dfrac{1}{6}$ ***AG***

[2 marks]

(b) $G(t) = \dfrac{1}{3}t + \dfrac{2}{3} \times \dfrac{1}{4}t^2 + \dfrac{2}{3} \times \dfrac{3}{4} \times \dfrac{1}{3}t^3 + \dfrac{2}{3} \times \dfrac{3}{4} \times \dfrac{2}{3} \times \dfrac{1}{4}t^4 + \ldots$ ***M1A1***

$= \dfrac{1}{3}t\left(1 + \dfrac{1}{2}t^2 + \ldots\right) + \dfrac{1}{6}t^2\left(1 + \dfrac{1}{2}t^2 + \ldots\right)$ ***M1A1***

$= \dfrac{\frac{t}{3}}{1 - \frac{t^2}{2}} + \dfrac{\frac{t^2}{6}}{1 - \frac{t^2}{2}}$ ***A1A1***

$= \dfrac{2t + t^2}{6 - 3t^2}$ ***AG***

[6 marks]

(c) $G'(t) = \dfrac{(2 + 2t)(6 - 3t^2) + 6t(2t + t^2)}{(6 - 3t^2)^2}$ ***M1A1***

$\mathrm{E}(X) = G'(1) = \dfrac{10}{3}$ ***M1A1***

[4 marks]

Total [12 marks]

3. (a) we are given that $M \sim \mathrm{N}(30, 9)$ and $F \sim \mathrm{N}(20, 6.25)$

let $X = M - 2F$; $X \sim \mathrm{N}(-10, 34)$ ***M1A1A1***

we require $\mathrm{P}(X > 0)$ ***(M1)***

$= 0.0432$ ***A1***

[5 marks]

(b) let $Y = M_1 + M_2 + F_1 + F_2 + F_3 + F_4 + F_5$; $Y \sim \mathrm{N}(160, 49.25)$ ***M1A1A1***

we require $\mathrm{P}(Y < 175) = 0.984$ ***A1***

[4 marks]

Total [9 marks]

4. (a) $\mathrm{H}_0 : \rho = 0$; $\mathrm{H}_1 : \rho > 0$ ***A1***

[1 mark]

(b) (i) correlation coefficient $= 0.905$ ***A2***

p-value $= 2.61 \times 10^{-5}$ ***A2***

(ii) very strong evidence to indicate a positive association between marks in Mechanics and marks in Statistics ***R1***

[5 marks]

(c) the regression line of y on x is $y = 8.71 + 0.789x$ ***(M1)A1***

George's estimated mark on Paper 2 $= 8.71 + 0.789 \times 63$ ***(M1)***

$= 58$ ***A1***

[4 marks]

(d) $t = r\sqrt{\dfrac{n-2}{1-r^2}} = 2.3019\ldots$ ***M1A1***

degrees of freedom $= 14$ ***(A1)***

p-value $= 0.0186\ldots$ ***A1***

at the 1 % significance level, this does not indicate a positive association between the marks in Physics and Chemistry ***R1***

[5 marks]

Total [15 marks]

5. (a) $\mathrm{E}(X) = 1 \times \theta + 2 \times 2\theta + 3(1 - 3\theta) = 3 - 4\theta$ ***M1A1***

$\mathrm{Var}(X) = 1 \times \theta + 4 \times 2\theta + 9(1 - 3\theta) - (3 - 4\theta)^2$ ***M1A1***

$= 6\theta - 16\theta^2$ ***AG***

[4 marks]

(b) (i) $\mathrm{E}(\hat{\theta}_1) = \dfrac{3 - \mathrm{E}(\overline{X})}{4} = \dfrac{3 - (3 - 4\theta)}{4} = \theta$ ***M1A1***

so $\hat{\theta}_1$ is an unbiased estimator of θ ***AG***

$\mathrm{Var}(\hat{\theta}_1) = \dfrac{6\theta - 16\theta^2}{16n}$ ***A1***

(ii) each of the n observed values has a probability θ of having the value 1 ***R1***

so $Y \sim \mathrm{B}(n, \theta)$ ***AG***

$\mathrm{E}(\hat{\theta}_2) = \dfrac{\mathrm{E}(Y)}{n} = \dfrac{n\theta}{n} = \theta$ ***A1***

$\mathrm{Var}(\hat{\theta}_2) = \dfrac{n\theta(1-\theta)}{n^2} = \dfrac{\theta(1-\theta)}{n}$ ***M1A1***

(iii) $\mathrm{Var}(\hat{\theta}_1) - \mathrm{Var}(\hat{\theta}_2) = \dfrac{6\theta - 16\theta^2 - 16\theta + 16\theta^2}{16n}$ ***M1***

$= \dfrac{-10\theta}{16n} < 0$ ***A1***

$\hat{\theta}_1$ is the more efficient estimator since it has the smaller variance ***R1***

[10 marks]

Total [14 marks]