# CAS and Dymanic Geometry Activities That Integrate Algebra and Geometry: Investigate, Discover, Prove

Tom Reardon

Fitch High School, 35 years

Youngstown State University, 34 years

"It is better to know how to learn than to know."

- Dr. Seuss

"With every mistake
we must surely be learning"

- George Harrison
"While My Guitar Gently Weeps"

# email addresses

http://bitly/TIME2014TR

# Integrating CAS into Algebra and Geometry

# Background:
# Summer math courses
# at Exeter Academy

## Ian Winokur   Square Problem

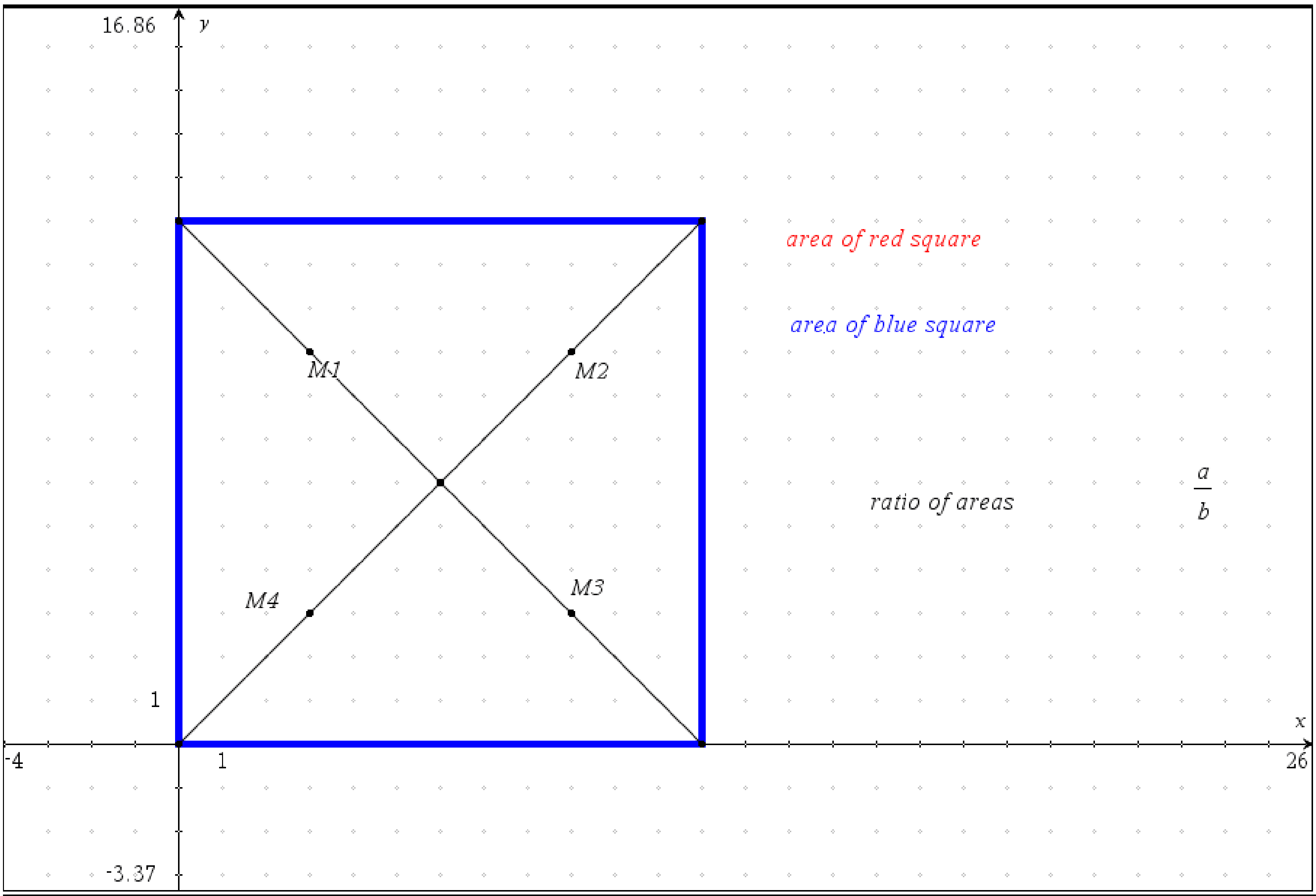
Given a square with both diagonals drawn.

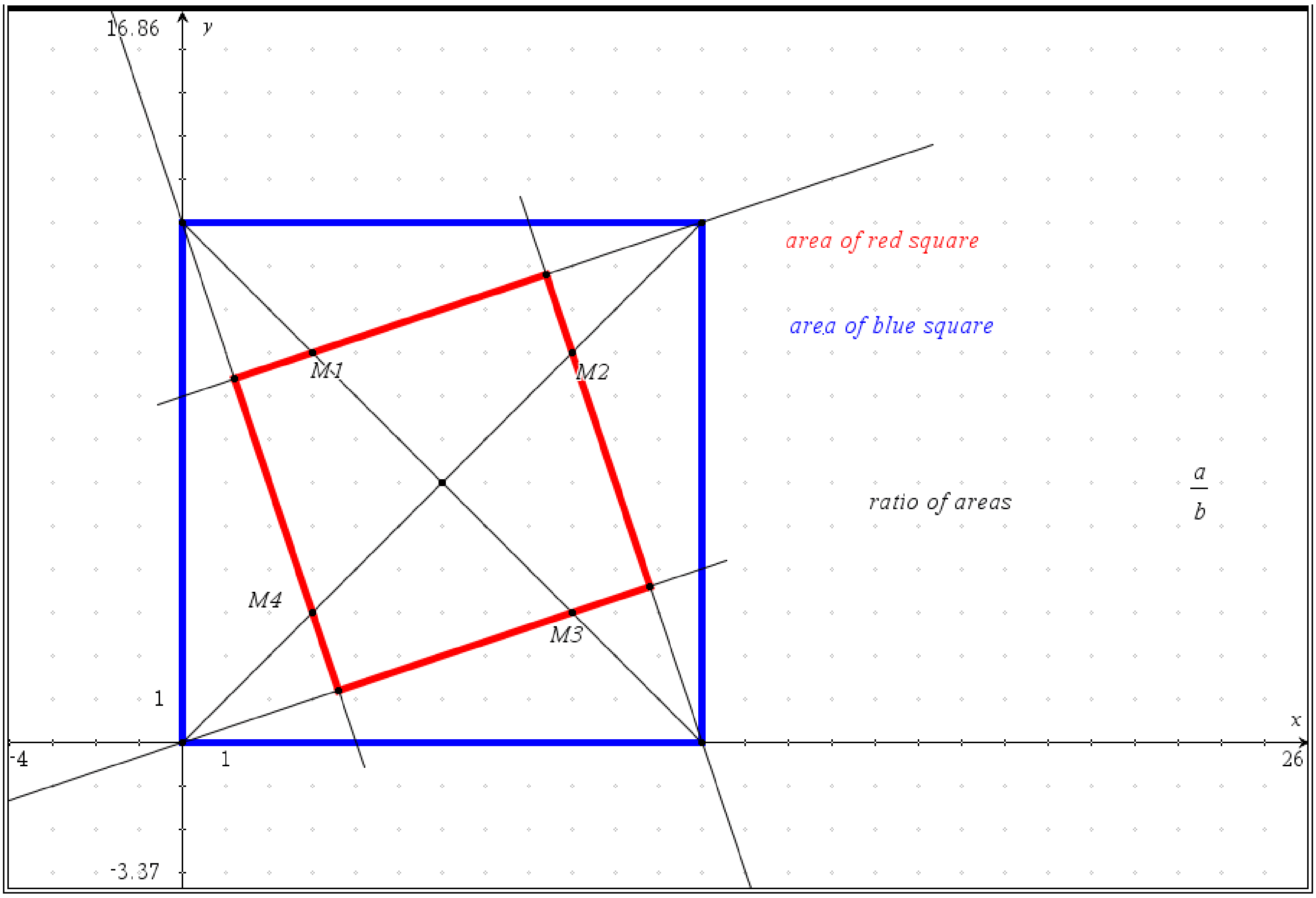
Find the midpoints of the 4 "half-diagonals".

Draw the 4 segments connecting each vertex with a different midpoint.

1. Of what type is the resulting figure?

2. What is the relationship between the area of the original square and the resulting figure?


16.86
y
area of red square
area of blue square
M1
M2
ratio of areas
a
b
M4
M3
1
x
-4
1
26
-3.37


16.86
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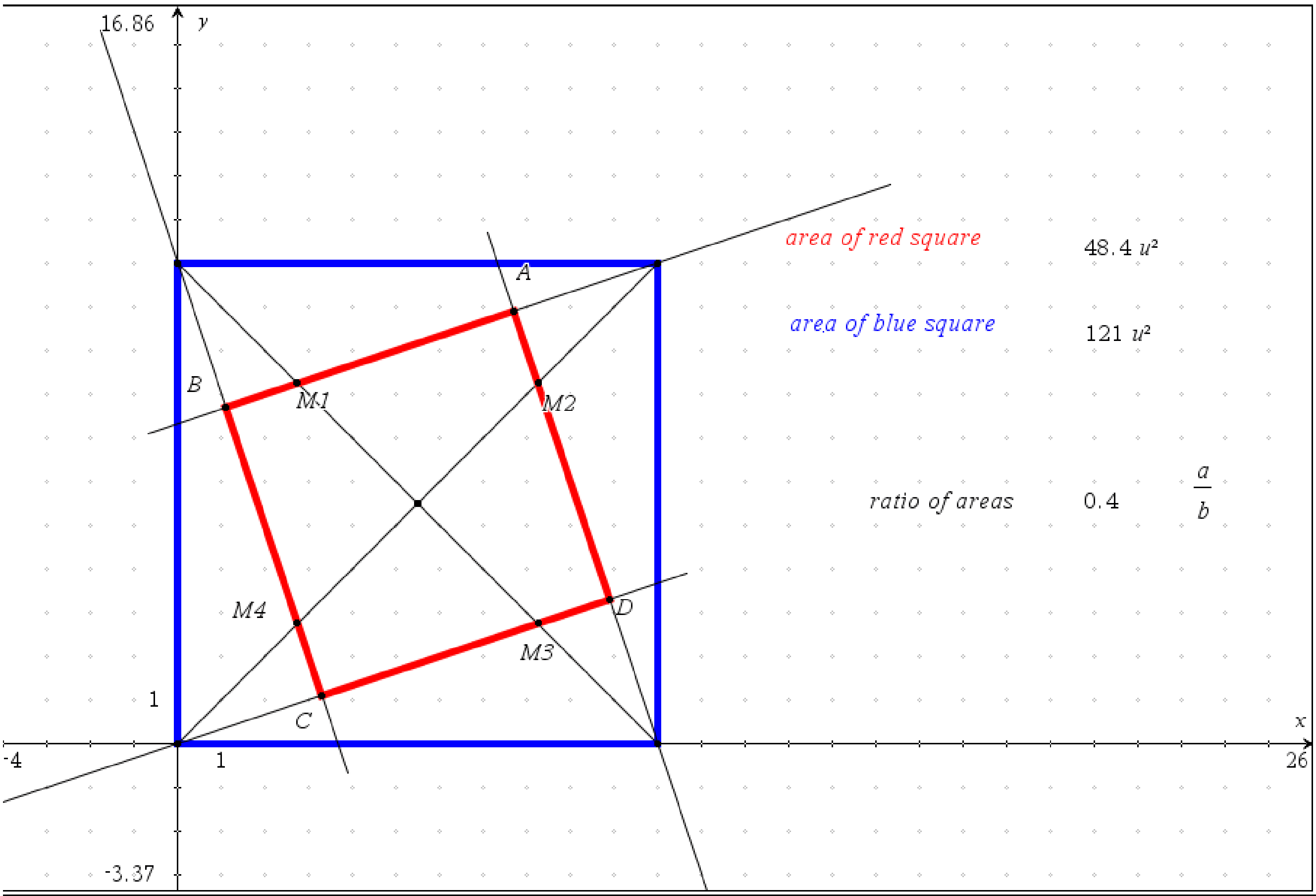
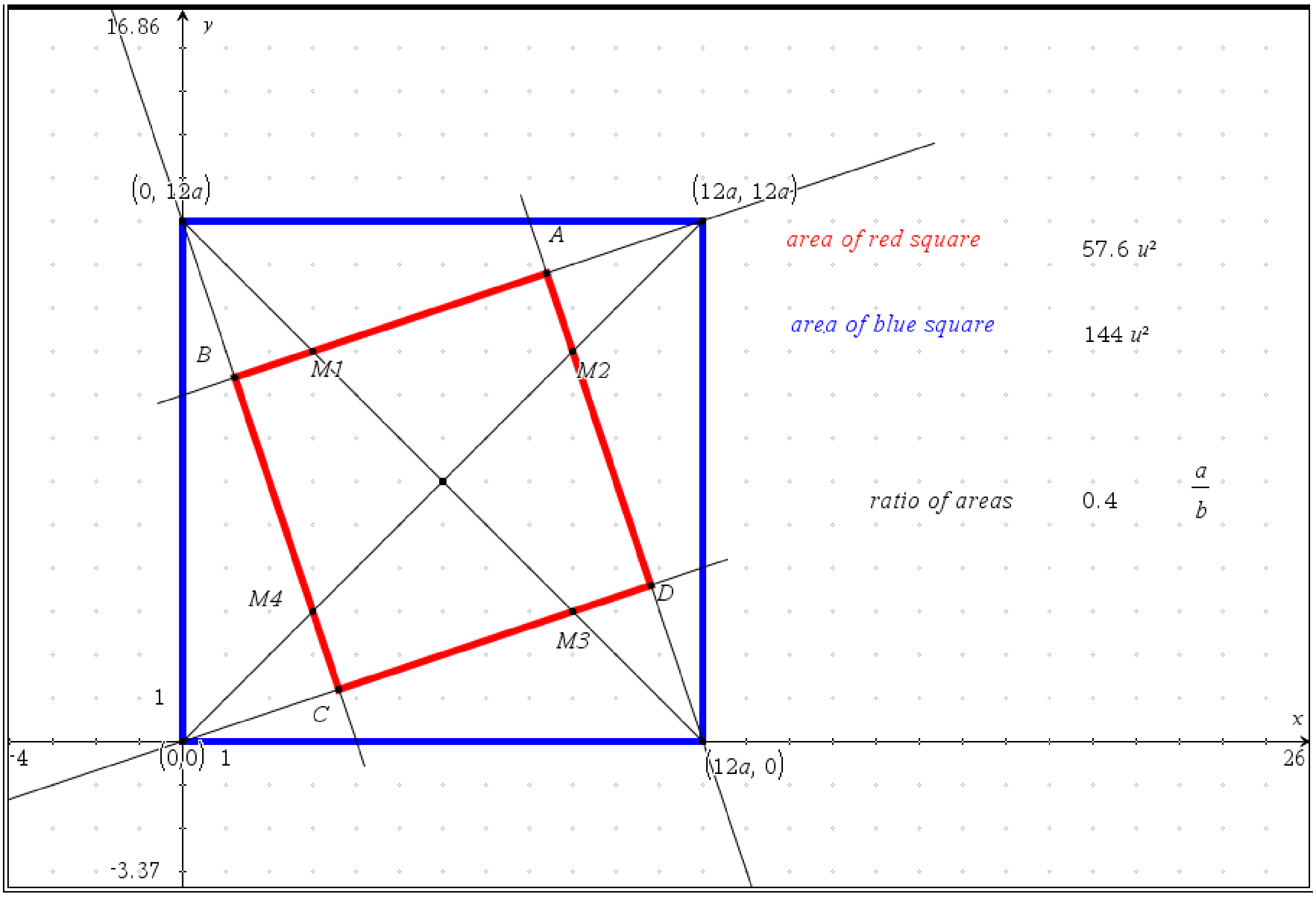

16.86
y
area of red square
48.4 u²
area of blue square
121 u²
A
B
M1
M2
ratio of areas
0.4
a/b
M4
D
M3
1
C
x
-4
1
26
-3.37


16.86
y
(0, 12a)
(12a, 12a)
A
area of red square
57.6 u²
area of blue square
144 u²
B
M1
M2
ratio of areas
0.4
a/b
M4
D
M3
1
C
x
-4
(0,0)
1
(12a, 0)
26
-3.37

| | |
|---|---|
| $\text{linSolve}\left(\begin{cases} y=\frac{1}{3}\cdot x+8\cdot a \\ y=-3\cdot x+12\cdot a \end{cases},\{x,y\}\right)$ | $\left\{\frac{6\cdot a}{5},\frac{42\cdot a}{5}\right\}$ |
| $\frac{6\cdot a}{5}\to bx$ | $\frac{6\cdot a}{5}$ |
| $\frac{42\cdot a}{5}\to by$ | $\frac{42\cdot a}{5}$ |
| $\frac{42\cdot a}{5}\to by$ | $\frac{42\cdot a}{5}$ |
| $\text{linSolve}\left(\begin{cases} y=-3\cdot x+12\cdot a \\ y=\frac{1}{3}\cdot x \end{cases},\{x,y\}\right)$ | $\left\{\frac{18\cdot a}{5},\frac{6\cdot a}{5}\right\}$ |
| $\frac{18\cdot a}{5}\to cx$ | $\frac{18\cdot a}{5}$ |
| $\frac{6\cdot a}{5}\to cy$ | $\frac{6\cdot a}{5}$ |
| $\text{linSolve}\left(\begin{cases} y=-3\cdot x+36\cdot a \\ y=\frac{1}{3}\cdot x \end{cases},\{x,y\}\right)$ | $\left\{\frac{54\cdot a}{5},\frac{18\cdot a}{5}\right\}$ |
| $\frac{54\cdot a}{5}\to ax$ | $\frac{54\cdot a}{5}$ |

16/99

© length of AB

$\sqrt{(ax-bx)^2+(ay-by)^2}$ $\frac{12\cdot |a|\cdot \sqrt{10}}{5}$

© length of BC

$\sqrt{(bx-cx)^2+(by-cy)^2}$ $\frac{12\cdot |a|\cdot \sqrt{10}}{5}$

© length of CD

$\sqrt{(cx-dx)^2+(cy-dy)^2}$ $\frac{12\cdot |a|\cdot \sqrt{10}}{5}$

© length of DA

$\sqrt{(dx-ax)^2+(dy-ay)^2}$ $\frac{12\cdot |a|\cdot \sqrt{10}}{5}$

© Therefore ABCD is equilateral

8/99

© Slope of AB

$$\frac{by-ay}{bx-ax} \qquad \frac{1}{3}$$

© Slope of BC

$$\frac{cy-by}{cx-bx} \qquad -3$$

© Slope of CD

$$\frac{dy-cy}{dx-cx} \qquad \frac{1}{3}$$

© Slope of DA

$$\frac{dy-ay}{dx-ax} \qquad -3$$

© Therefore opposite sides parallel; consecutive sides perpendicular;

8/99

© Equilateral quadrilateral with 4 right angles is a square

© Now calculate the area of the inside square

$$\left(\frac{12\cdot |a|\cdot \sqrt{10}}{5}\right)^2 \qquad \frac{288\cdot a^2}{5}$$

© Now calculate the area of the original square

$$(12\cdot a)^2 \qquad 144\cdot a^2$$

© Calculate the ratio of the area of the inside square to the area of the outside square

$$\frac{\frac{288\cdot a^2}{5}}{144\cdot a^2} \qquad \frac{2}{5}$$

© Therefore, the smaller square is 40% of the larger square

8/99

## Find the Distance from a Point to a Line Activity

*Using TI-Nspire CAS*
*C 2007 Reardon Gifts, Inc.*
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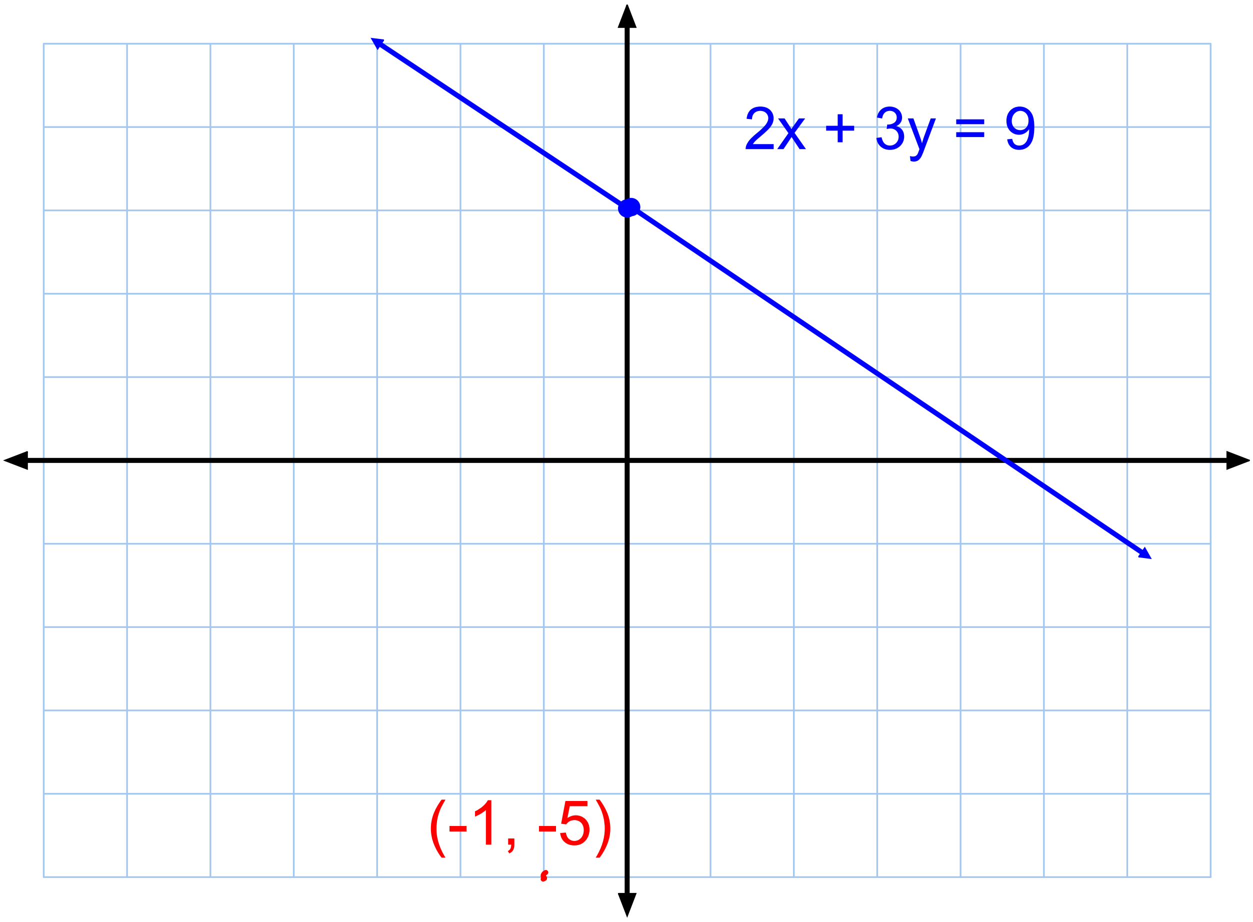
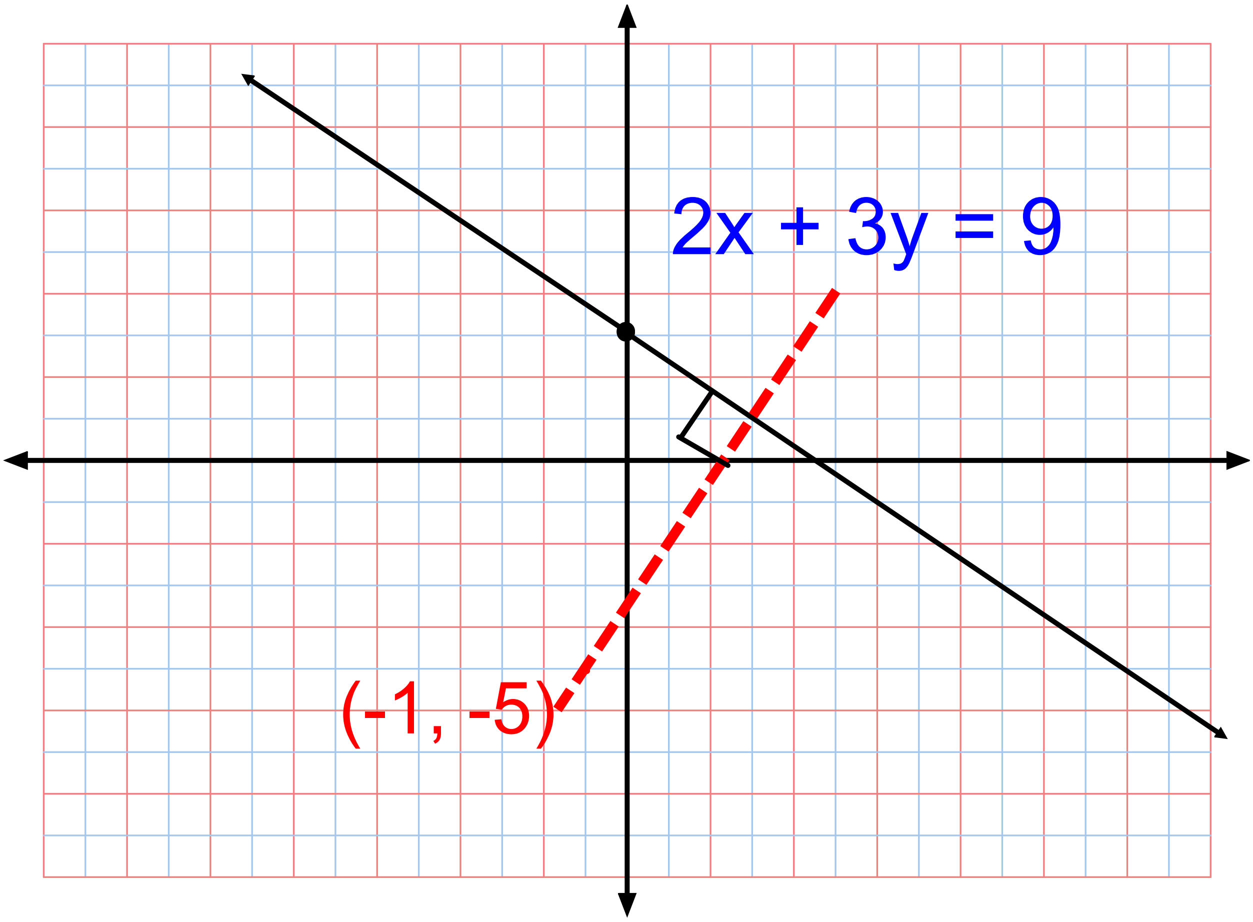
**Part 1 Specific Solution with no CAS**

A. Using graph paper, find the distance from the point (– 1, – 5) to the line 2x + 3y = 9. Show as much work as you can graphically. Compute your answer in both exact mode and rounded to the nearest thousandth.

B. Using TI-Nspire CAS, graphically find the distance from the point (– 1, – 5) to the line 2x + 3y = 9. Use the page labeled Part 1B of the TI-Nspire document "Distance from a Point to a Line" that is supplied to you. Compute your answer to the nearest thousandth and compare the Nspire solution with your answer to Part A. These answers should be the same. If not, attempt to make them the same answer.

C. Using the graphical solution to assist you, find the distance from the point (– 1, – 5) to the line 2x + 3y = 9, but show all the parts of the solution algebraically. Do this "by hand", that is, do not use the CAS features of the TI-Nspire CAS to assist you. Compute your answer in both exact mode and rounded to the nearest thousandth.

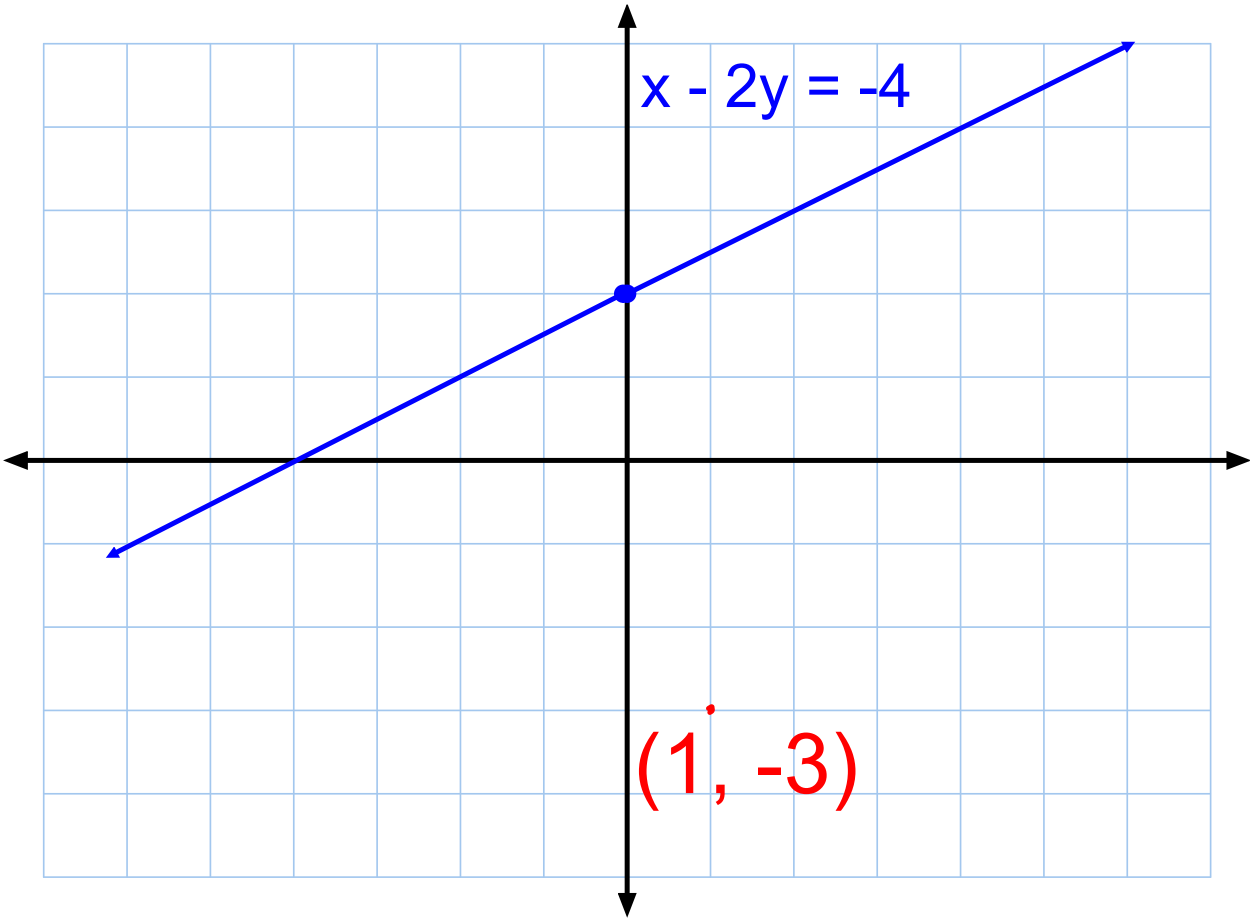

2x + 3y = 9
(-1, -5)

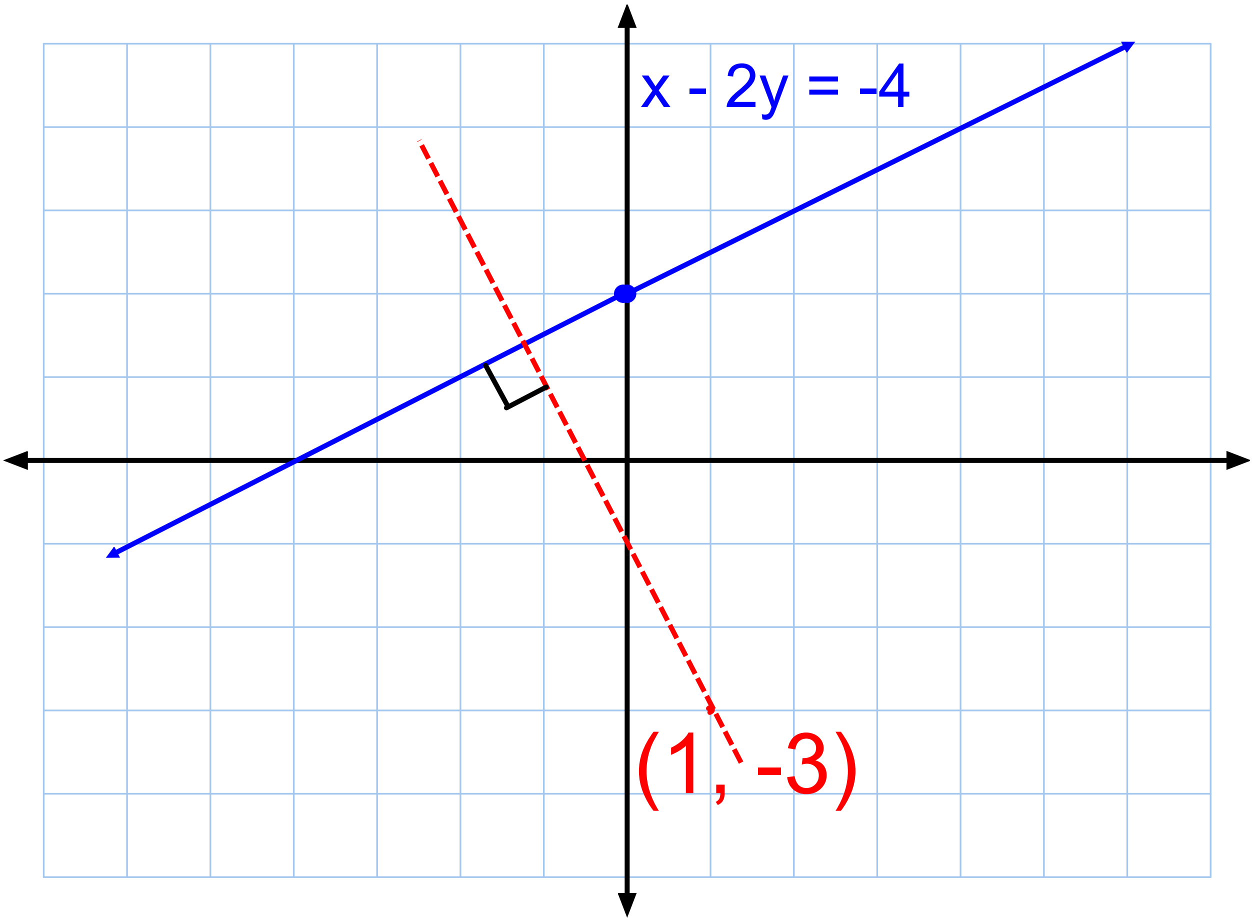

2x + 3y = 9
(-1, -5)

**Part 2 Specific Solution with CAS assistance**

D. Using the knowledge and techniques learned from Parts A, B,and C (above), algebraically find the distance between the point (1, – 3) to the line x – 2y = – 4. Use the CAS features of the TI-Nspire CAS to assist you as needed. Record intermediate results so that anyone can follow your reasoning. Keep your CAS steps in your TI-Nspire CAS document on the page labeled Part 2D. Compute your answer in both exact mode and rounded to the nearest thousandth.

E. Using TI-Nspire CAS, graphically find the distance from the point (1, – 3) to the line x – 2y = – 4. Use the page labeled Part 2E. Compute your answer to the nearest thousandth and compare the Nspire solution with your answer to Part D. These answers should be the same. If not, attempt to make them the same answer.


x - 2y = -4
(1, -3)


x - 2y = -4
(1, -3)

**Part 3 General Solution with CAS assistance**

F. Using the knowledge and techniques learned from Parts D and E (above), algebraically find the distance between the point (x1, y1) to the line $a \cdot x + b \cdot y = c$. Use the CAS features of the TI-Nspire CAS to assist you as needed. Record intermediate results so that anyone can follow your reasoning. Keep your CAS steps in your TI-Nspire CAS document on the page labeled Part 3F.

G. Using whatever resources you have, find the actual formula for calculating the distance from a point to a line. Compare this to the answer you obtained in Part F above.

Find the distance from the point P (x1, y1) to ax +by =c.

1. Find the slope of the original line

ax + by = c

slope of this line $-\frac{a}{b}$

slope of line perpendicular to this line $\frac{b}{a}$

2. Find the equation of the line perpendicular to the given line through the given point.

$$y - y1 = \frac{b}{a}(x - x1)$$

3. Solve the system:

$$y - y1 = \frac{b}{a}(x - x1)$$

$$a \cdot x + b \cdot y = c$$

| | |
|---|---|
| $y-y1=\frac{b}{a}\cdot(x-x1)\to e1$ | $y-y1=\frac{b\cdot(x-x1)}{a}$ |
| $a\cdot x+b\cdot y=c\to e2$ | $a\cdot x+b\cdot y=c$ |
| $\text{linSolve}\left(\begin{cases}e1\\e2\end{cases},\{x,y\}\right)$ | $\left\{\left\{\frac{-(a\cdot(b\cdot y1-c)-b^2\cdot x1)}{a^2+b^2},a\neq0,\left\{\frac{a^2\cdot y1-a\cdot b\cdot x1+b\cdot c}{a^2+b^2},a\neq0\right.\right.\right.$ ▸ |
| $\frac{-(a\cdot(b\cdot y1-c)-b^2\cdot x1)}{a^2+b^2}\to x$ | $\frac{-(a\cdot(b\cdot y1-c)-b^2\cdot x1)}{a^2+b^2}$ |
| $\frac{a^2\cdot y1-a\cdot b\cdot x1+b\cdot c}{a^2+b^2}\to y$ | $\frac{a^2\cdot y1-a\cdot b\cdot x1+b\cdot c}{a^2+b^2}$ |

4. Find the distance between 2 points:
the solution to the system and the given point.

$\sqrt{(x1-xs)^2+(y1-ys)^2}$

$\frac{|a\cdot x1+b\cdot y1-c|}{\sqrt{a^2+b^2}}$

http://en.wikipedia.org/wiki/Distance_from_a_point_to_a_line

In the case of a line in the plane given by the equation $ax + by + c = 0$, where $a$, $b$ and $c$ are real constants with $a$ and $b$ not both zero, the distance from the line to a point $(x_0, y_0)$ is

$$\operatorname{distance}(ax + by + c = 0, (x_0, y_0)) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

# Midpoint Polygon Investigation

# Midpoint Polygons

## An Investigation

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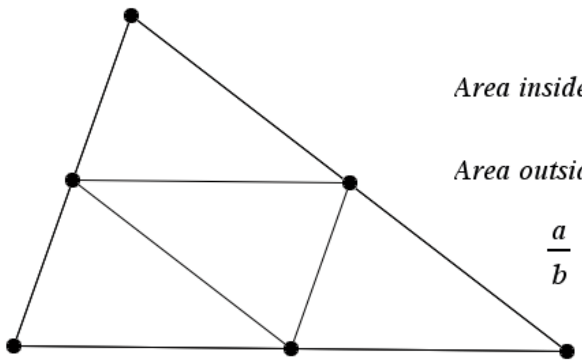
Investigate

Investigate

Investigate ...

1 cm

*Generic Triangle*



*Area inside triangle* = 15.6 $cm^2$

*Area outside triangle* =62.4 $cm^2$

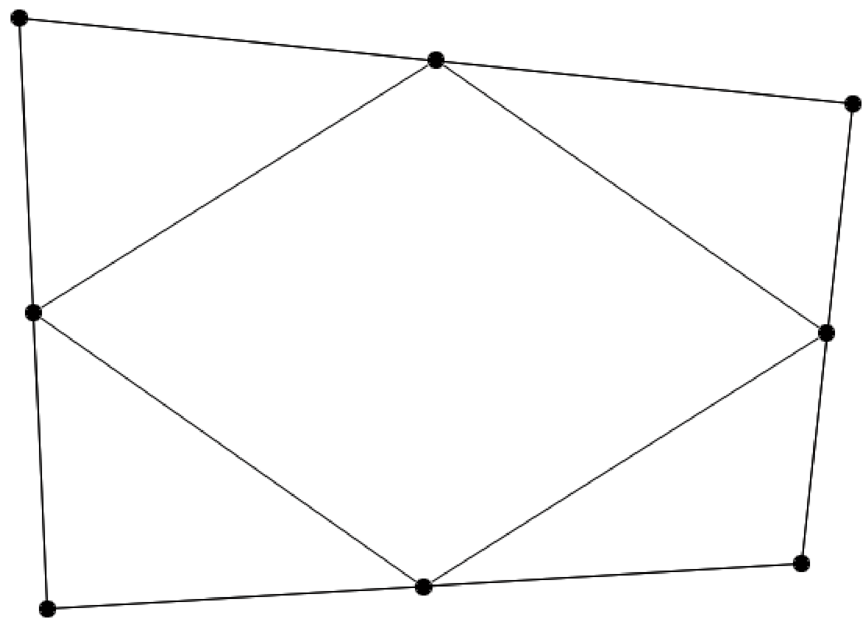
$\frac{a}{b}$ =0.25 = *ratio of areas*

*Generic Quadrilateral*

*Area of inside quad* = 109 $cm^2$

*Area of outside quad* = 217 $cm^2$

*ratio of areas* = 0.5

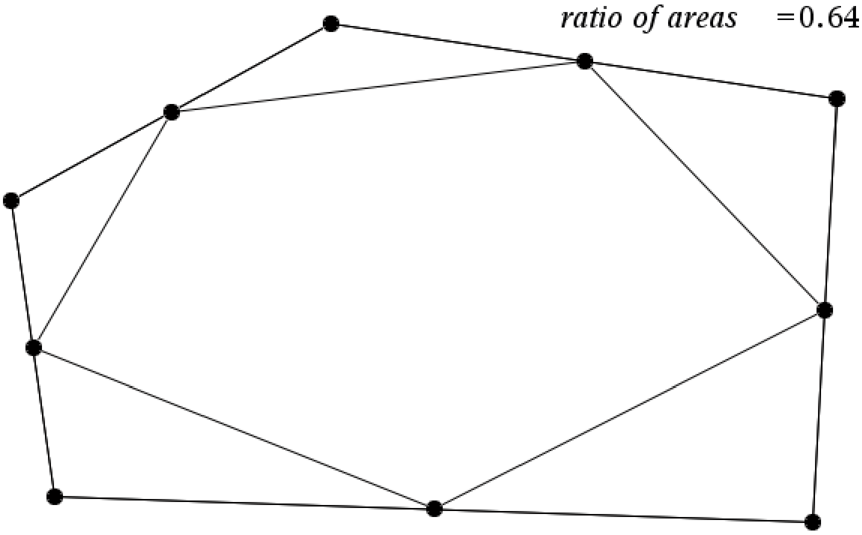


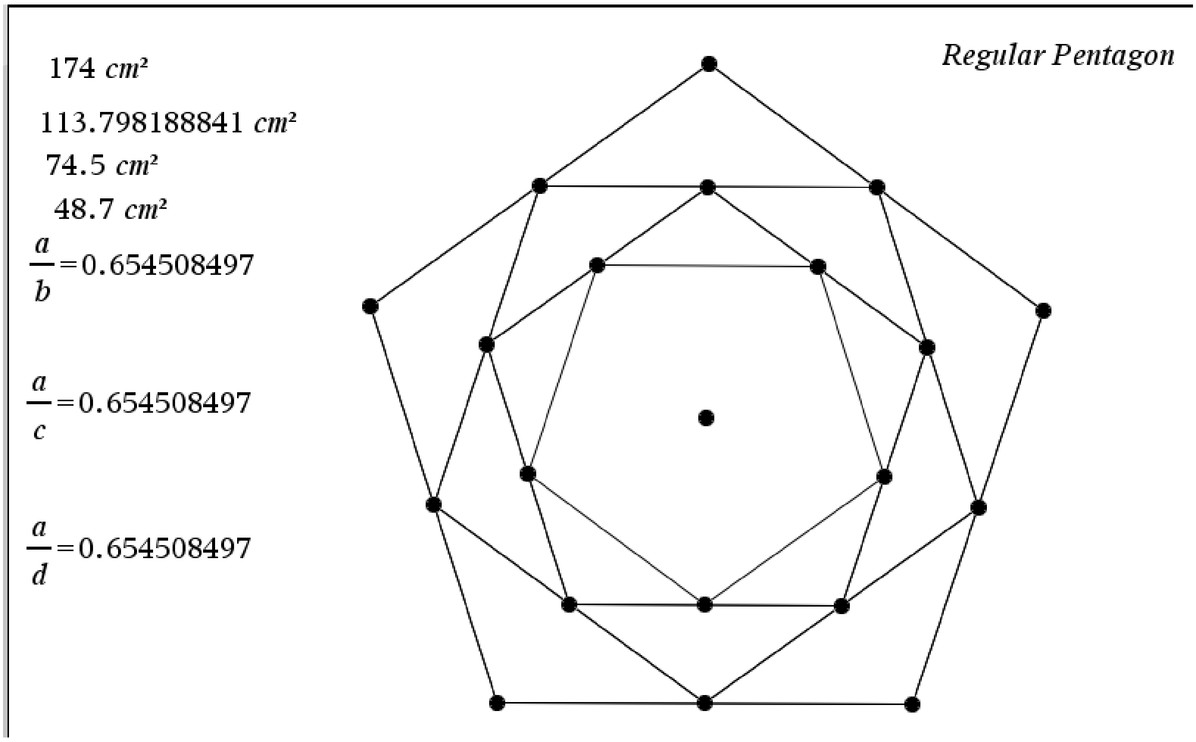
*Generic Pentagon*

*Area of inside pentagon =* 123 $cm^2$

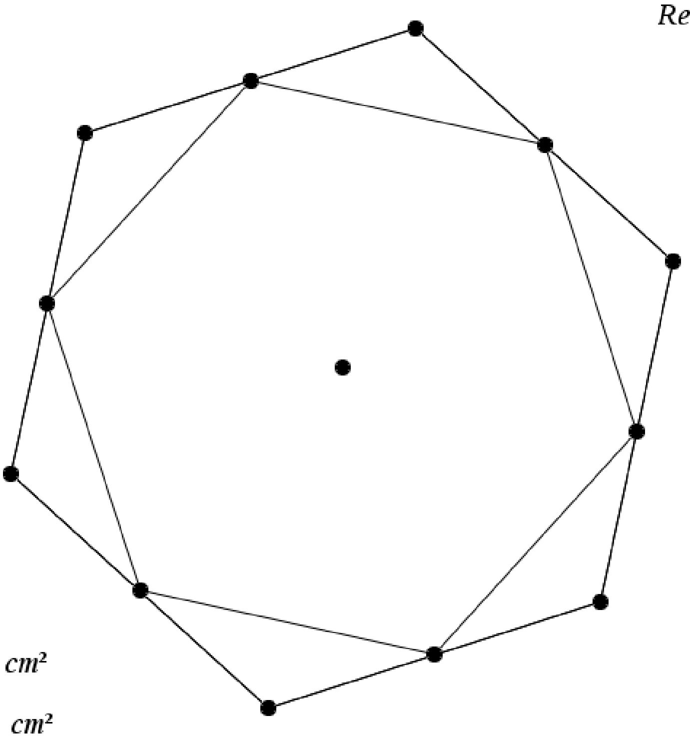
*Area of outside pentagon =* 192 $cm^2$

*ratio of areas* =0.642930899




Regular Pentagon
174 cm²
113.798188841 cm²
74.5 cm²
48.7 cm²
$\frac{a}{b} = 0.654508497$
$\frac{a}{c} = 0.654508497$
$\frac{a}{d} = 0.654508497$

*Regular Hexagon*

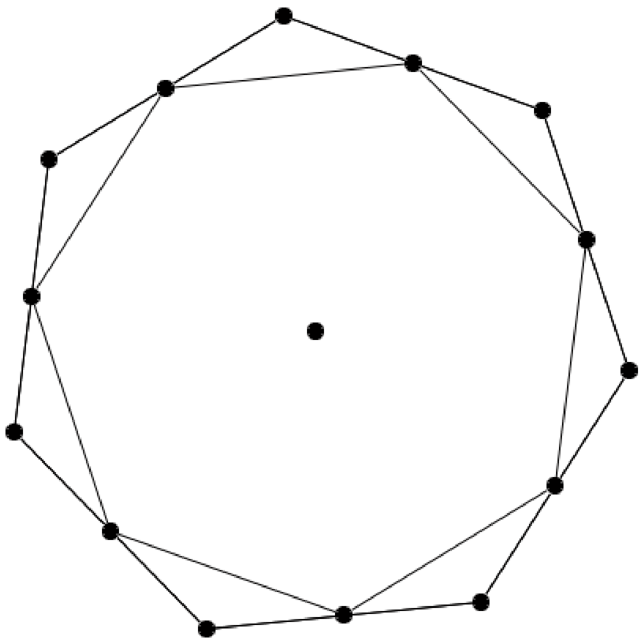


*Area inside* = 141 $cm^2$

*Area outside* = 188 $cm^2$

*Ratio* = 0.75

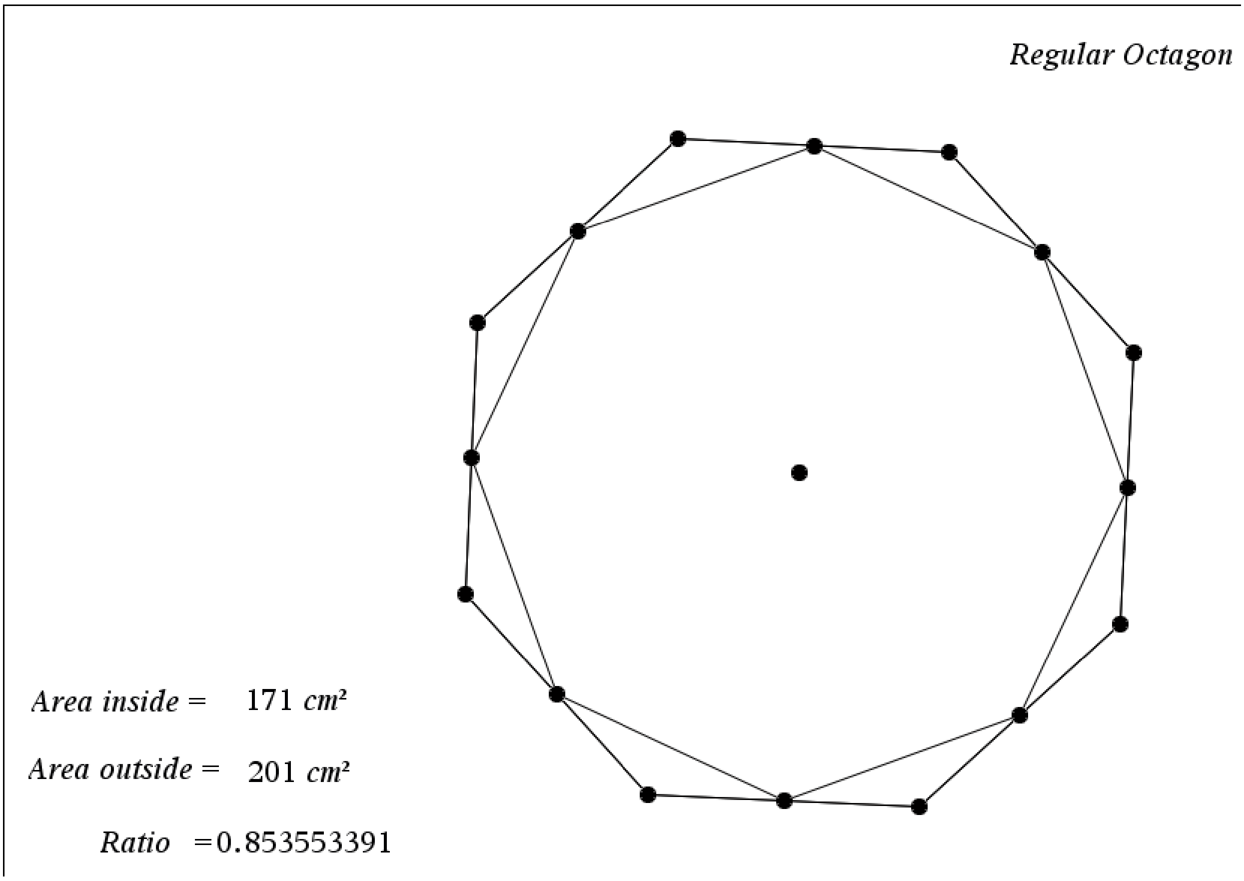
*Regular Heptagon*



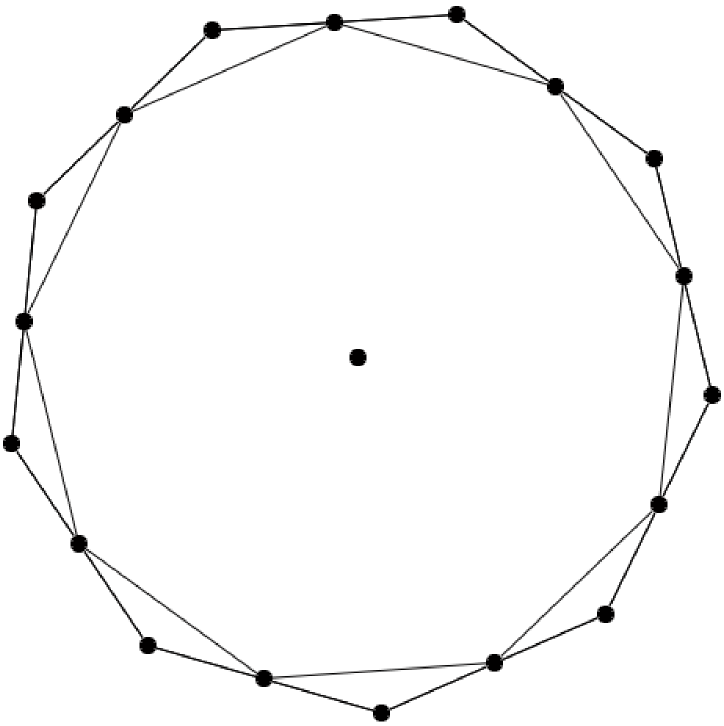
*Area inside* = 113 $cm^2$

*Area outside* = 140 $cm^2$

*ratio* = 0.811744901


Regular Octagon
Area inside = 171 cm²
Area outside = 201 cm²
Ratio =0.853553391

*Regular Nonagon*



*Area inside =* 168 $cm^2$

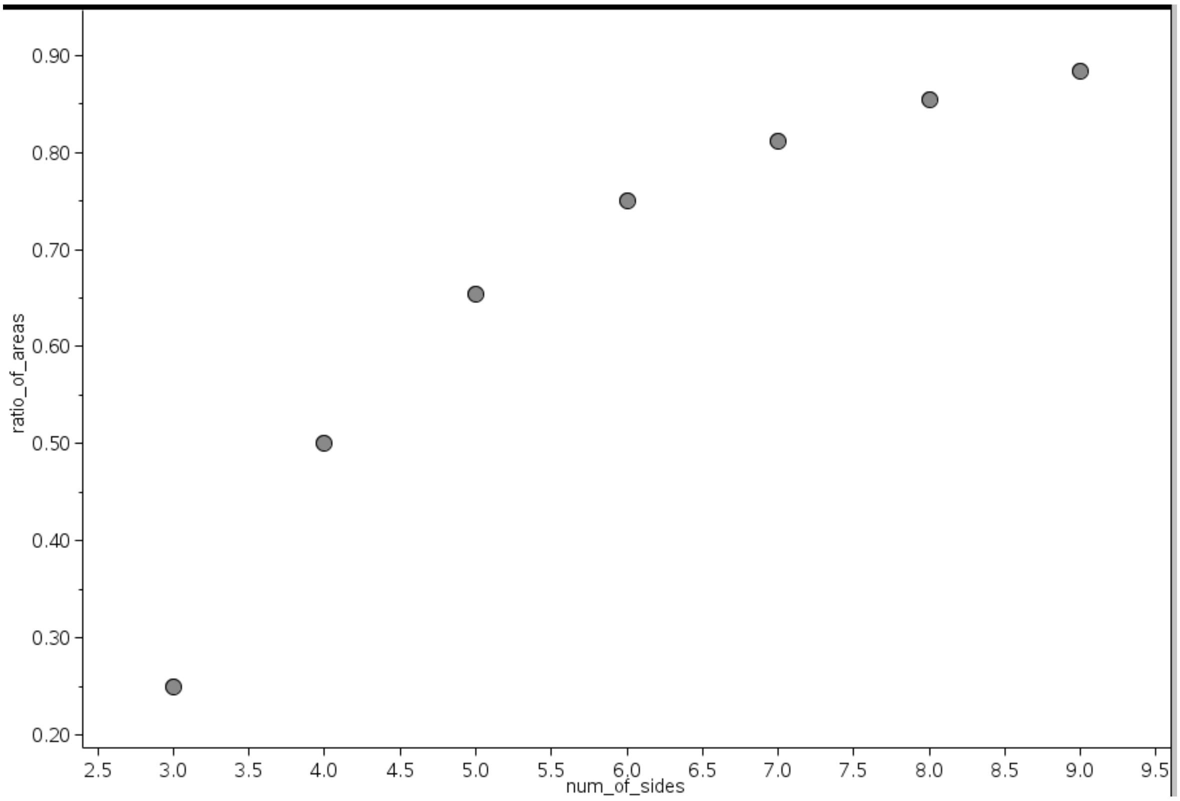
*Area outside =* 190 $cm^2$

*Ratio =* 0.883022222

Is there a pattern?

If so, what is it?

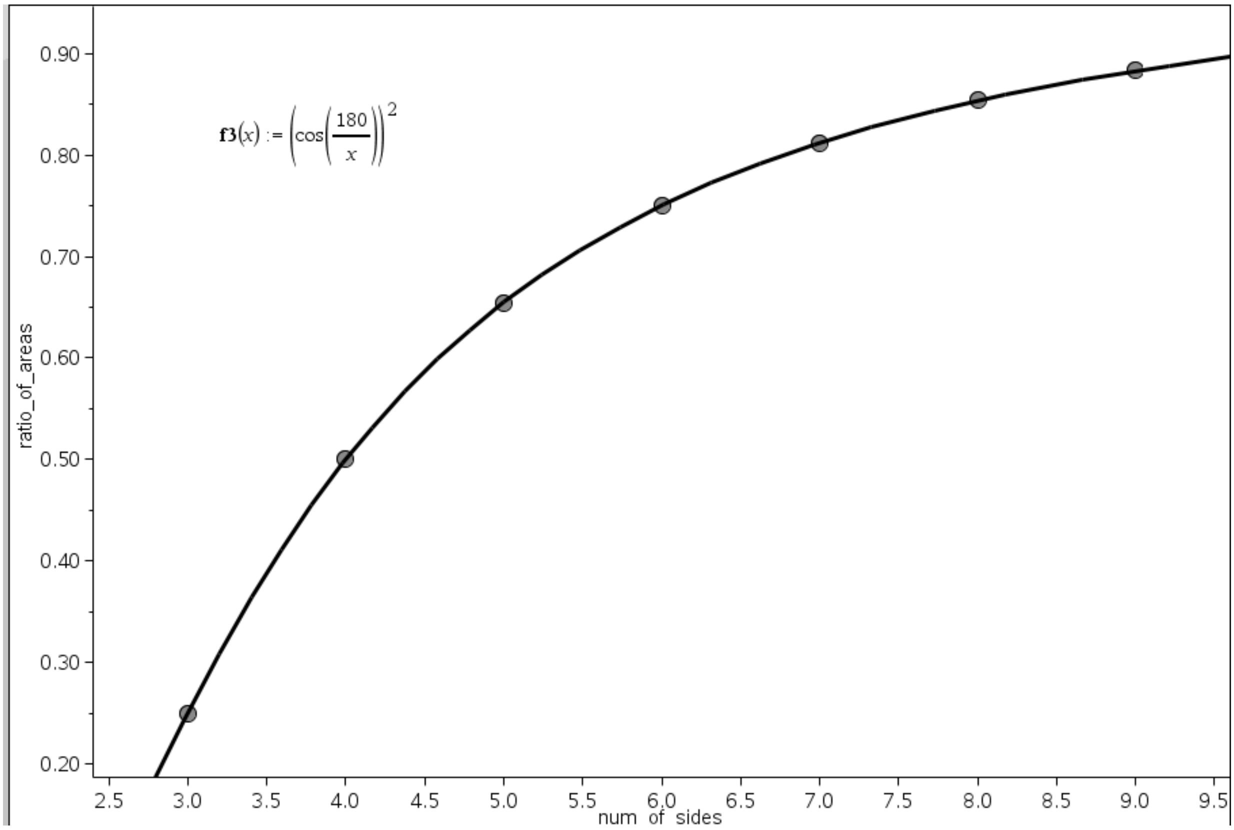
| | A num_of_sides | B ratio_of_areas |
|---|---|---|
| = | | |
| 1 | 3 | 0.250000000 |
| 2 | 4 | 0.500000000 |
| 3 | 5 | 0.654508497 |
| 4 | 6 | 0.750000000 |
| 5 | 7 | 0.811744901 |
| 6 | 8 | 0.853553391 |
| 7 | 9 | 0.883022222 |


0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
ratio_of_areas
2.5 3.0 3.5 4.0 4.5 5.0 5.5 6.0 6.5 7.0 7.5 8.0 8.5 9.0 9.5
num_of_sides

Is there a pattern?

If so, what is it?

The answer is on the next page.



$f3(x) := \left(\cos\left(\frac{180}{x}\right)\right)^2$
ratio_of_areas
num_of_sides
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
2.5
3.0
3.5
4.0
4.5
5.0
5.5
6.0
6.5
7.0
7.5
8.0
8.5
9.0
9.5

For a regular $n$-sided polygon.
The ratio of the inner area to the outer area is given by:

$$\left(\cos\left(\frac{180}{n}\right)\right)^2$$

**Explain why...**

| | A num_of_sides | B ratio_of_areas | C cos_formula | D |
|---|---|---|---|---|
| = | | | =(cos(180./(a[])))^2 | |
| 1 | 3 | 0.250000000 | 0.250000000 | |
| 2 | 4 | 0.500000000 | 0.500000000 | |
| 3 | 5 | 0.654508497 | 0.654508497 | |
| 4 | 6 | 0.750000000 | 0.750000000 | |
| 5 | 7 | 0.811744901 | 0.811744901 | |
| 6 | 8 | 0.853553391 | 0.853553391 | |
| 7 | 9 | 0.883022222 | 0.883022222 | |
| 8 | | | | |

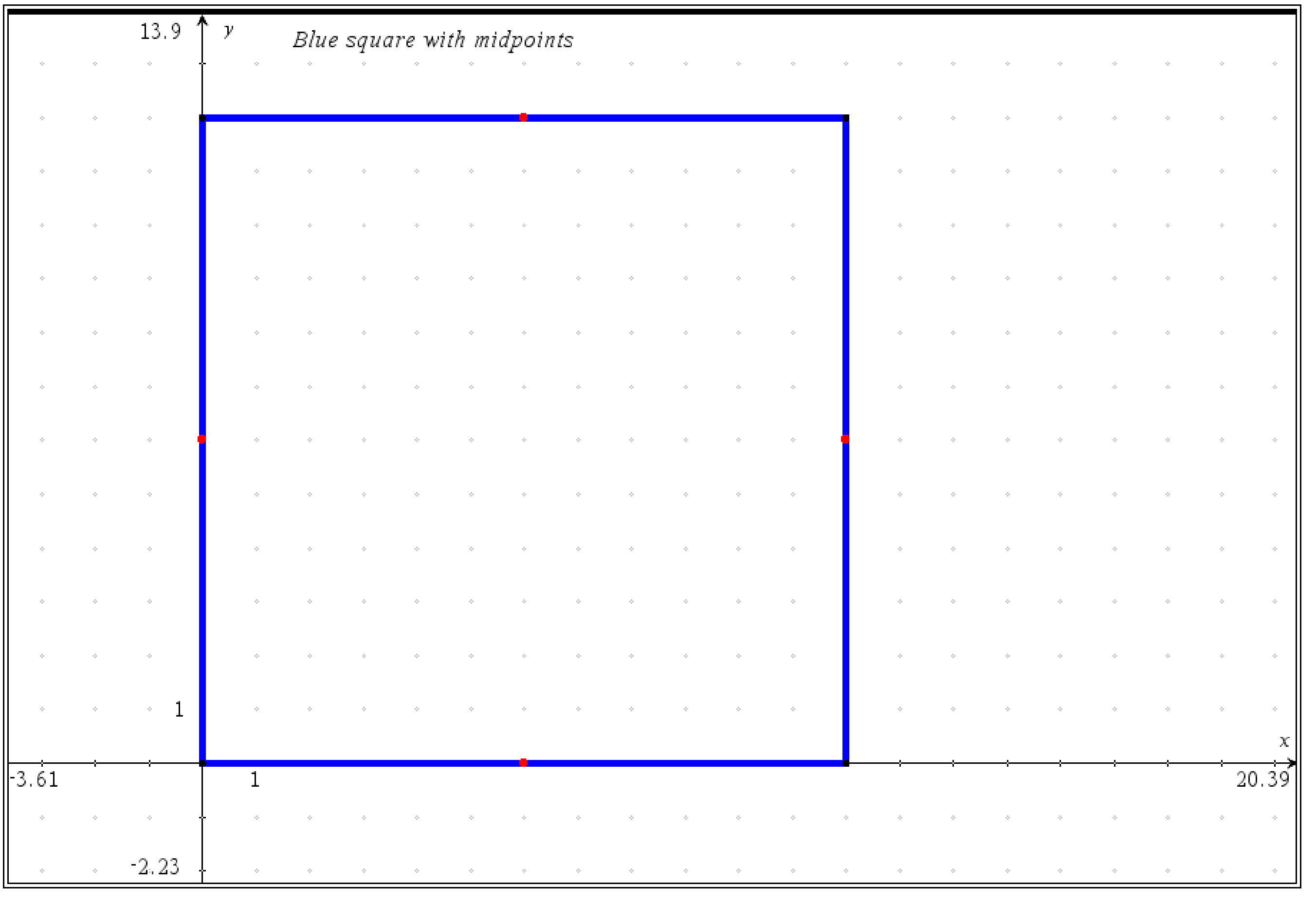
## Carmel Schittino Square Problem

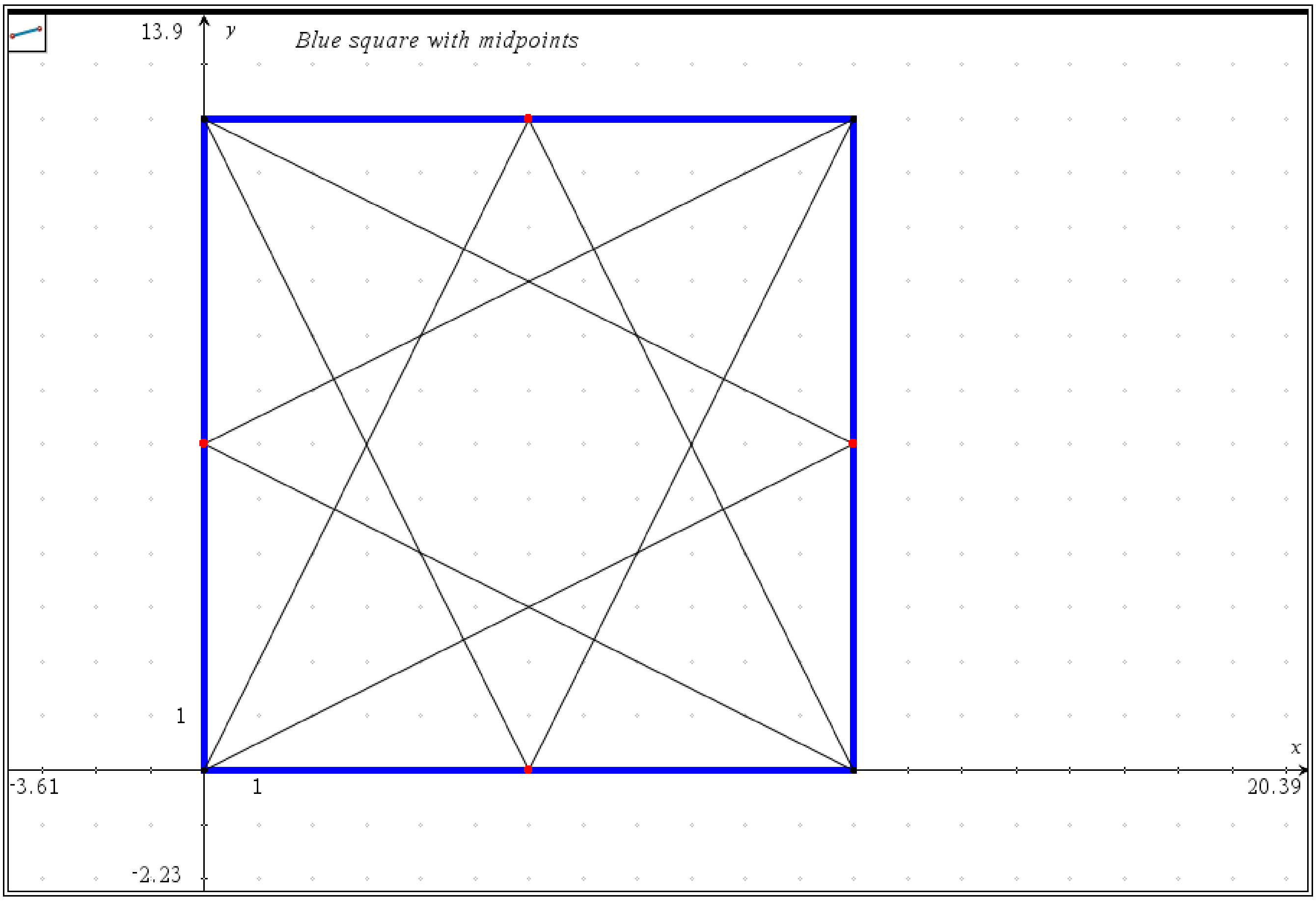
Given a square. Construct the midpoints of each side. Draw segments connecting each vertex to the 2 midpoints not on the sides that contain the vertex.

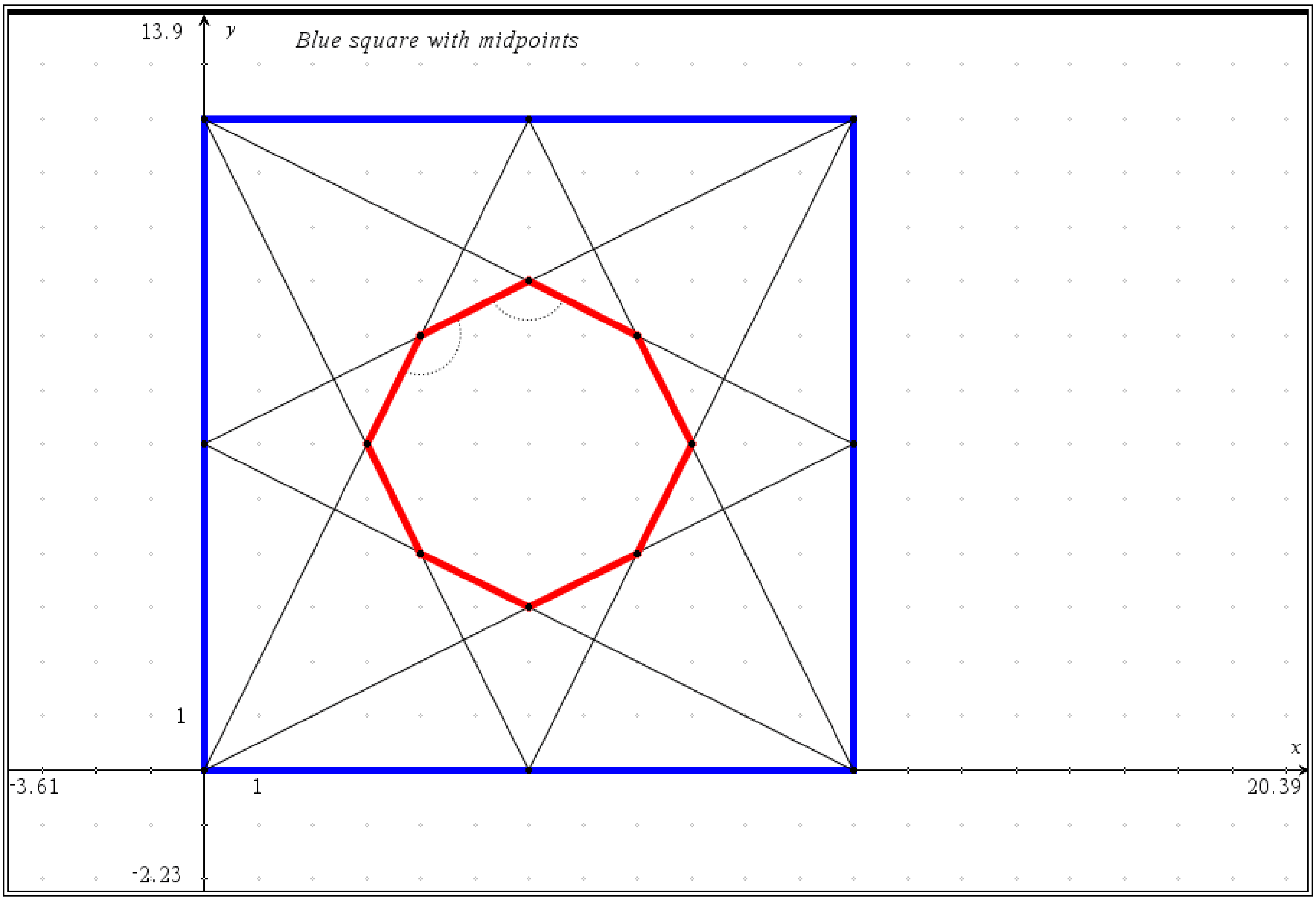
1. What is the best descriptor of the type of polygon that is formed.

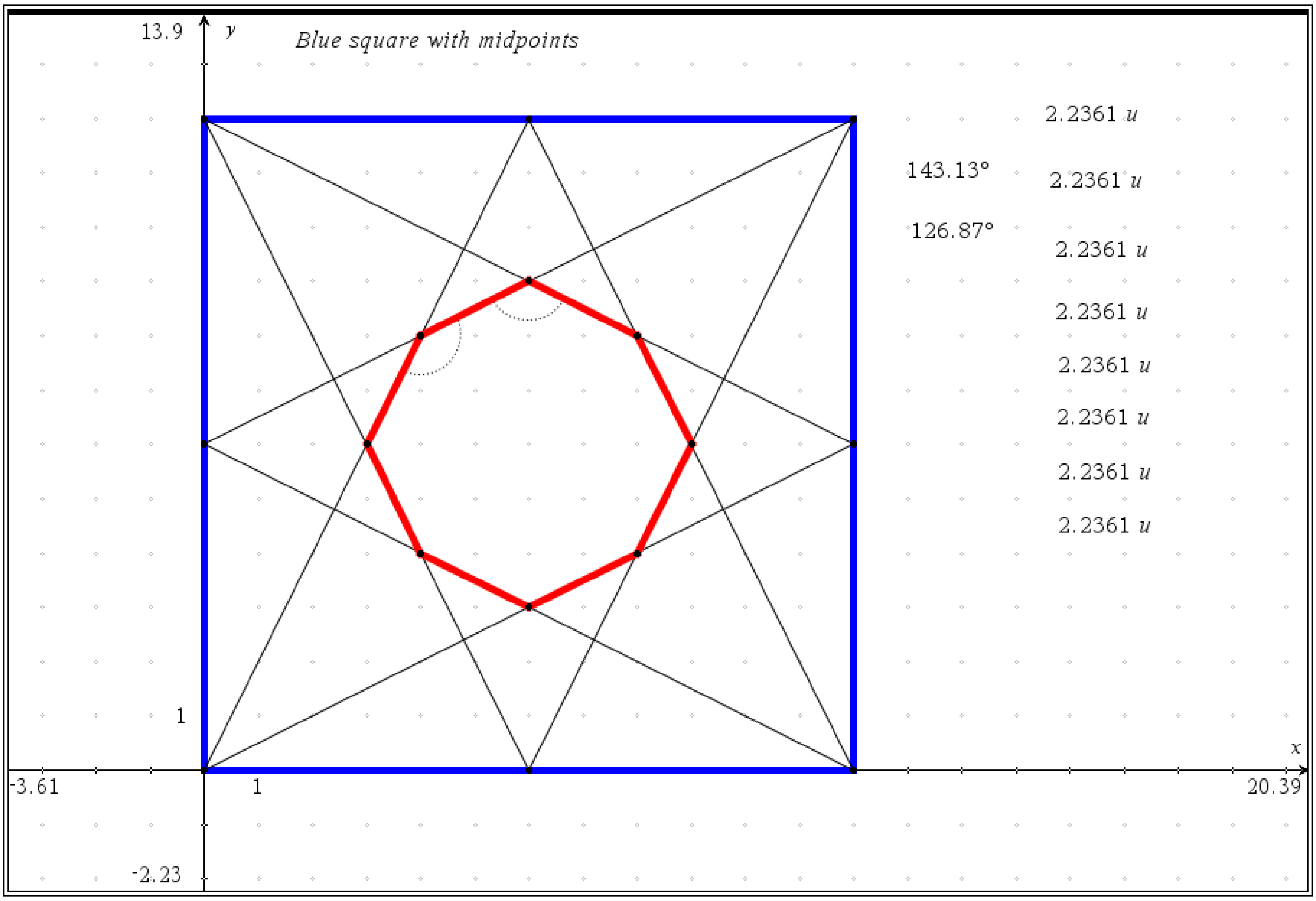
2. Is there a relationship between the area of the square the the area of the polygon?

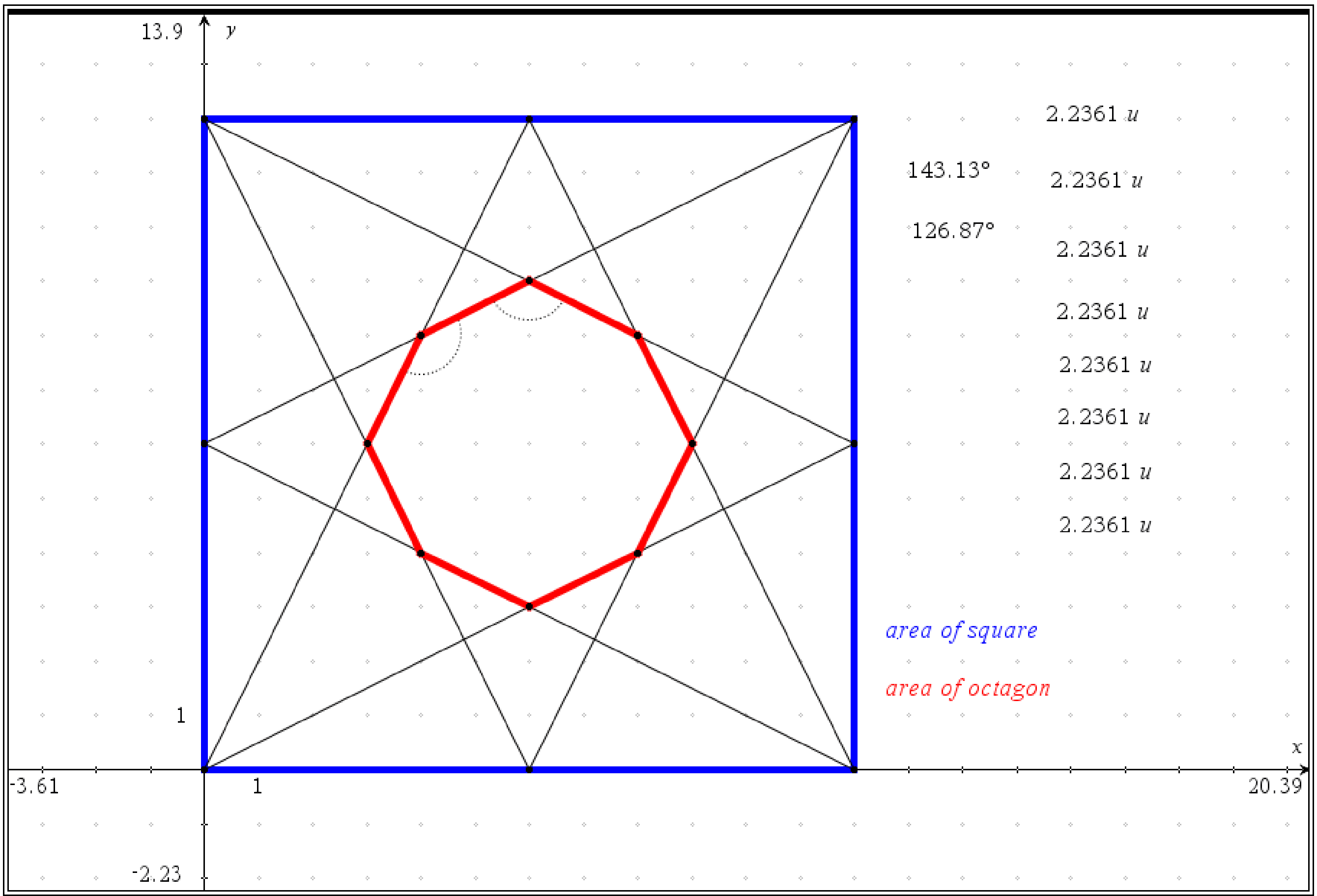
If so, what is it?

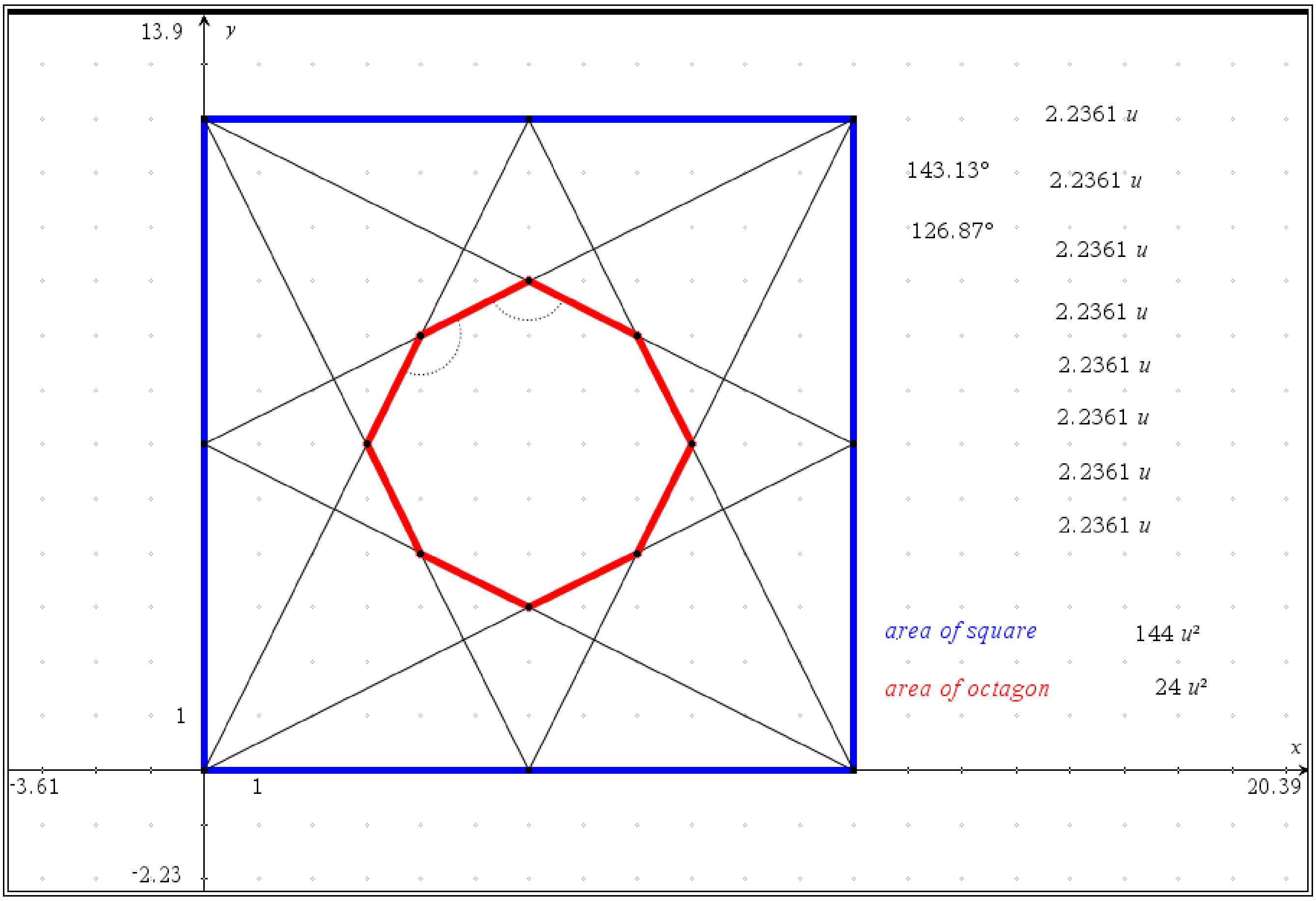

13.9
y
Blue square with midpoints
1
-3.61
1
20.39
x
-2.23

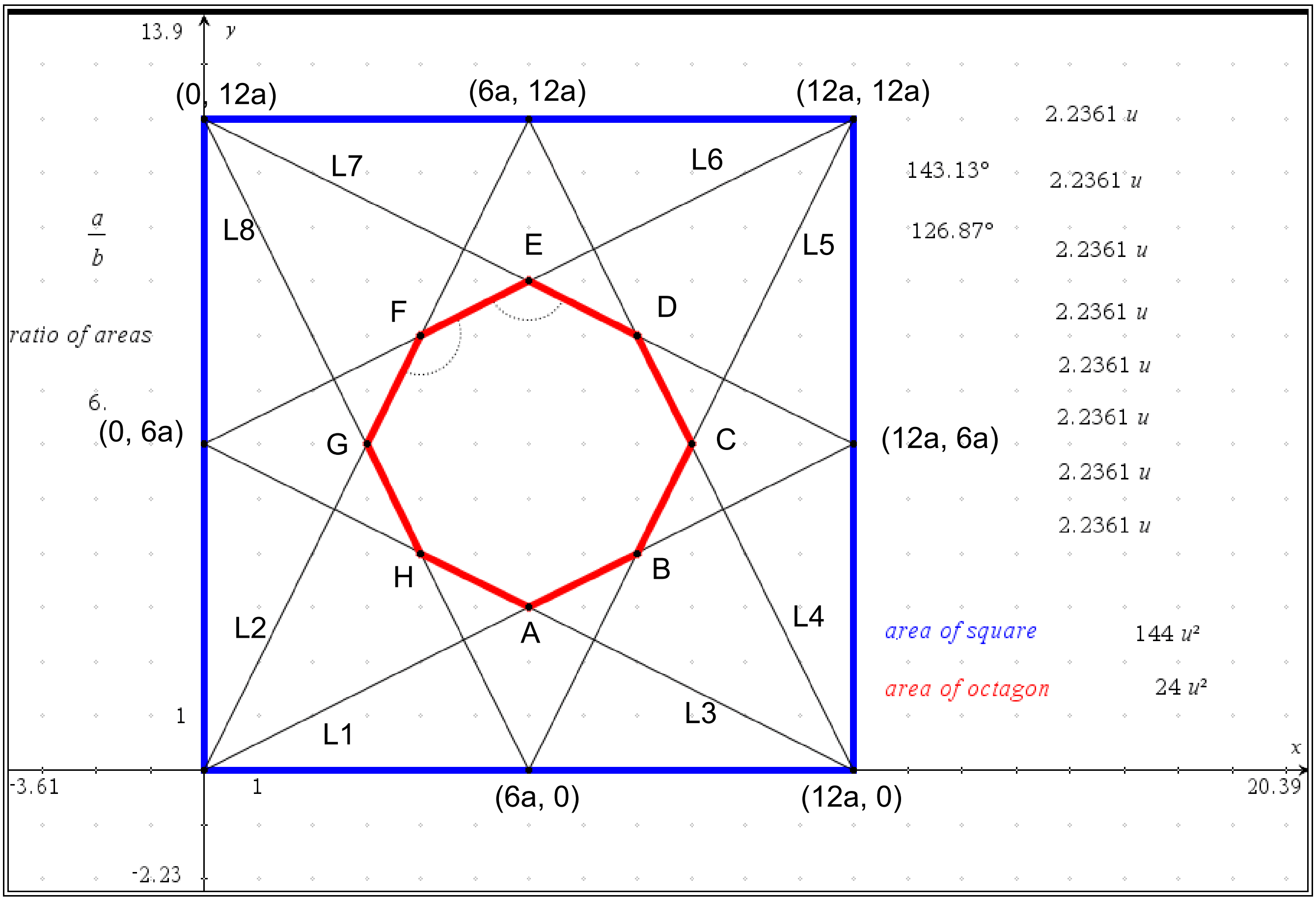

13.9
y
Blue square with midpoints
1
x
-3.61
1
20.39
-2.23


13.9
y
Blue square with midpoints
1
x
-3.61
1
20.39
-2.23

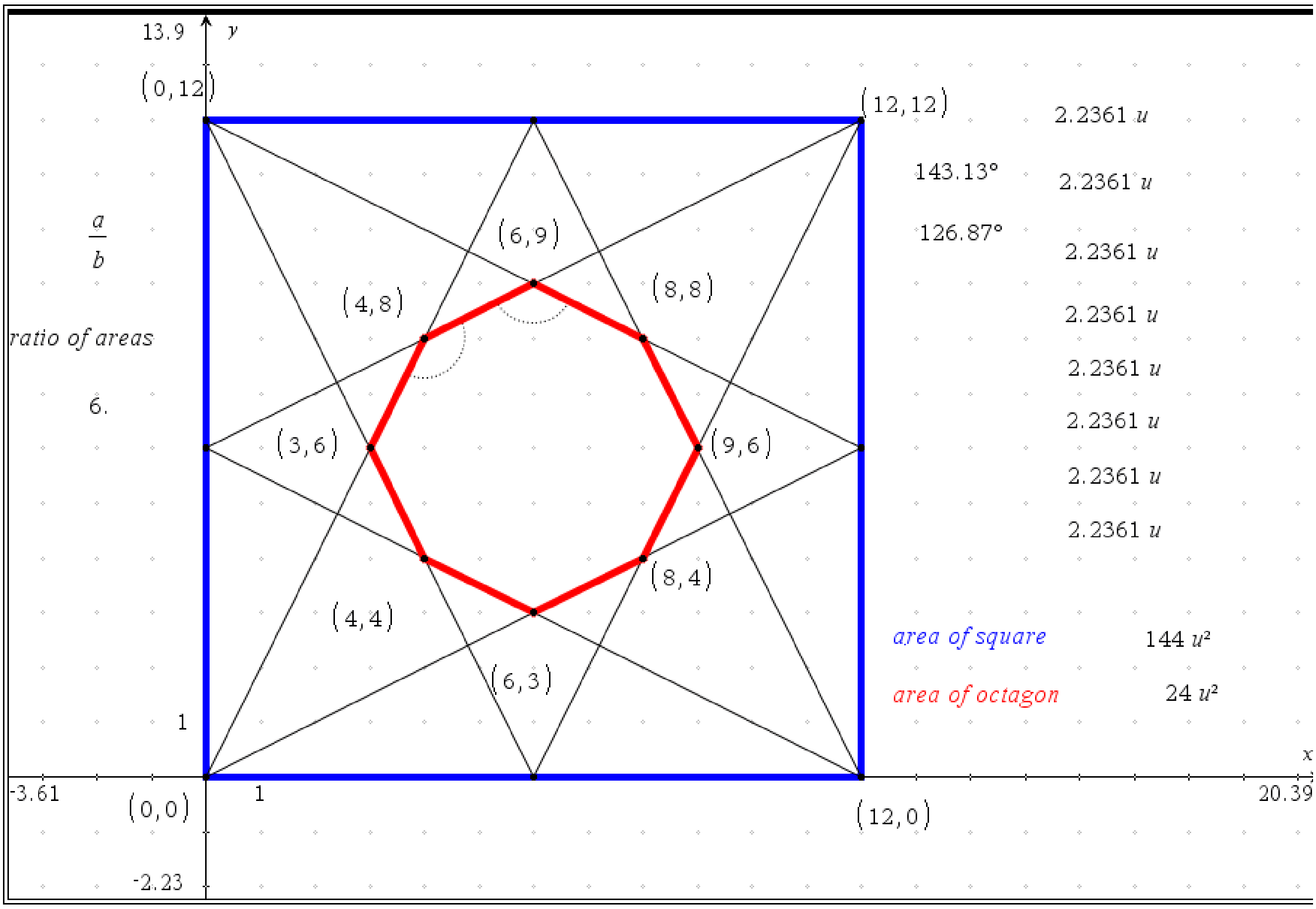

13.9
y
Blue square with midpoints
2.2361 u
143.13°
2.2361 u
126.87°
2.2361 u
2.2361 u
2.2361 u
2.2361 u
2.2361 u
2.2361 u
1
x
-3.61
1
20.39
-2.23


13.9
y
2.2361 u
143.13°
2.2361 u
126.87°
2.2361 u
2.2361 u
2.2361 u
2.2361 u
2.2361 u
2.2361 u
area of square
area of octagon
1
x
-3.61
1
20.39
-2.23


13.9
y
2.2361 u
143.13°
2.2361 u
126.87°
2.2361 u
2.2361 u
2.2361 u
2.2361 u
2.2361 u
2.2361 u
area of square
144 u²
area of octagon
24 u²
1
x
-3.61
1
20.39
-2.23

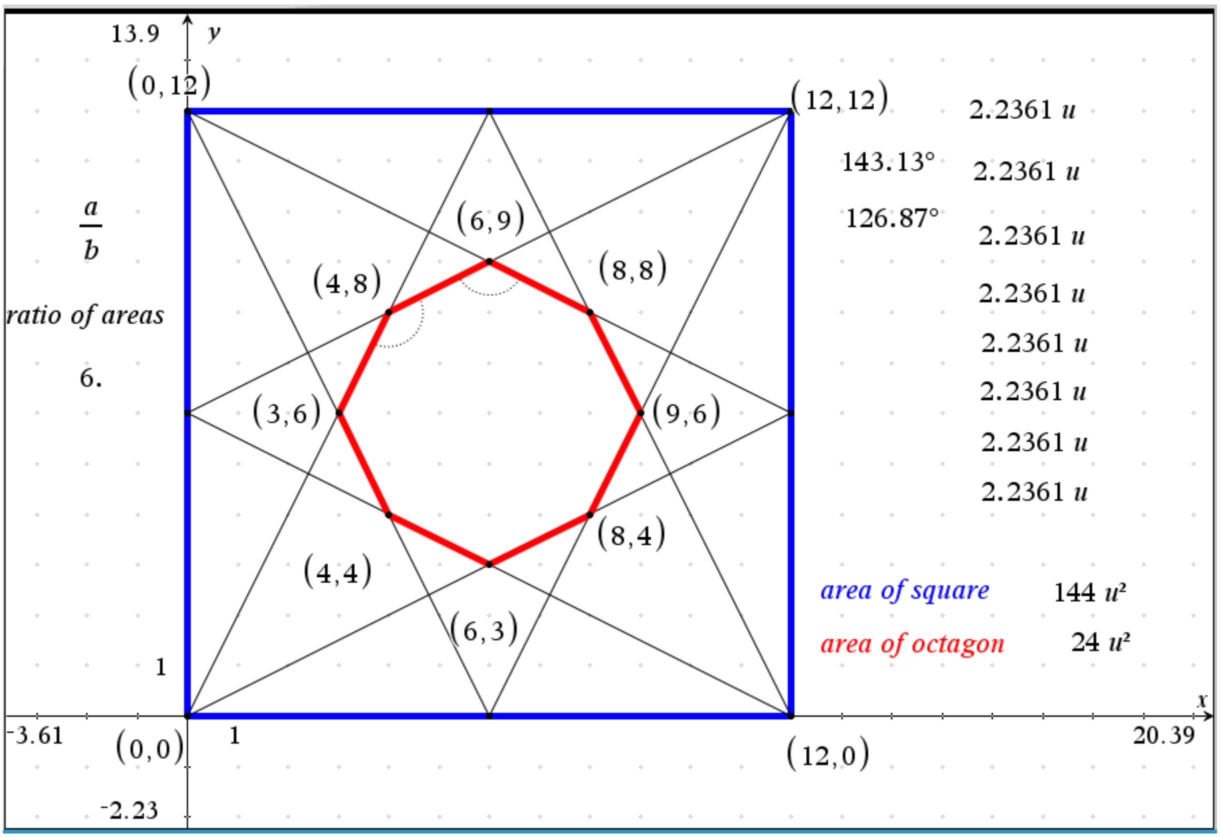


13.9
y
(0, 12a)
(6a, 12a)
(12a, 12a)
2.2361 u
L7
L6
143.13°
2.2361 u
a
b
L8
E
L5
126.87°
2.2361 u
F
D
2.2361 u
ratio of areas
2.2361 u
6.
2.2361 u
(0, 6a)
G
C
(12a, 6a)
2.2361 u
2.2361 u
H
B
L2
A
L4
area of square
144 u²
area of octagon
24 u²
1
L1
L3
x
-3.61
1
(6a, 0)
(12a, 0)
20.39
-2.23


13.9
y
(0,12)
(12,12)
2.2361 u
143.13°
2.2361 u
a/b
126.87°
2.2361 u
(6,9)
(4,8)
(8,8)
2.2361 u
ratio of areas
2.2361 u
6.
(3,6)
(9,6)
2.2361 u
2.2361 u
2.2361 u
(8,4)
(4,4)
area of square
144 u²
(6,3)
area of octagon
24 u²
1
x
-3.61
1
20.39
(0,0)
(12,0)
-2.23

| | |
|---|---|
| $y=\frac{1}{2}\cdot x \to ab$ | $y=\frac{x}{2}$ |
| $y=2\cdot x \to gj$ | $y=2\cdot x$ |
| $y=\frac{-1}{2}\cdot x+6\cdot a \to ha$ | $y=6\cdot a-\frac{x}{2}$ |
| $y=-2\cdot x+24\cdot a \to dc$ | $y=24\cdot a-2\cdot x$ |
| $y=2\cdot x-12\cdot a \to bc$ | $y=2\cdot x-12\cdot a$ |
| $y=\frac{1}{2}\cdot x+6\cdot a \to fe$ | $y=\frac{x}{2}+6\cdot a$ |
| $y=\frac{-1}{2}\cdot x+12\cdot a \to ea$ | $y=12\cdot a-\frac{x}{2}$ |
| $y=-2\cdot x+12\cdot a \to gh$ | $y=12\cdot a-2\cdot x$ |

| | |
|---|---|
| linSolve$\left(\begin{cases}ha\\ab\end{cases},\{x,y\}\right)\to a$ | $\{6\cdot a,3\cdot a\}$ |
| linSolve$\left(\begin{cases}bc\\ab\end{cases},\{x,y\}\right)\to b$ | $\{8\cdot a,4\cdot a\}$ |
| linSolve$\left(\begin{cases}bc\\dc\end{cases},\{x,y\}\right)\to c$ | $\{9\cdot a,6\cdot a\}$ |
| linSolve$\left(\begin{cases}dc\\ed\end{cases},\{x,y\}\right)\to d$ | $\{8\cdot a,8\cdot a\}$ |
| linSolve$\left(\begin{cases}ed\\fe\end{cases},\{x,y\}\right)\to e$ | $\{6\cdot a,9\cdot a\}$ |
| linSolve$\left(\begin{cases}fe\\gf\end{cases},\{x,y\}\right)\to f$ | $\{4\cdot a,8\cdot a\}$ |
| linSolve$\left(\begin{cases}gf\\gh\end{cases},\{x,y\}\right)\to g$ | $\{3\cdot a,6\cdot a\}$ |
| linSolve$\left(\begin{cases}gh\\ha\end{cases},\{x,y\}\right)\to h$ | $\{4\cdot a,4\cdot a\}$ |


13.9
y
(0, 12)
(12, 12)
2.2361 u
143.13°
2.2361 u
a/b
126.87°
2.2361 u
(6, 9)
(4, 8)
(8, 8)
2.2361 u
ratio of areas
2.2361 u
6.
2.2361 u
(3, 6)
(9, 6)
2.2361 u
2.2361 u
(8, 4)
(4, 4)
area of square
144 u²
(6, 3)
area of octagon
24 u²
1
x
-3.61
(0, 0)
1
(12, 0)
20.39
-2.23

"Mathematics is the garment that we continuously alter with our students, and technology should be seamlessly interwoven throughout its fabric."

Website with information from both of my talks at TIME 2014:

http://bit.ly/TIME2014TR

USA CAS conference July, 2015  Date TBD


(Peace)

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