The derivation of Kepler's three laws using Newton´s law of gravity and the law of force

A topic

in Mathematical Physics Seminar

at Secondary School

Introduction

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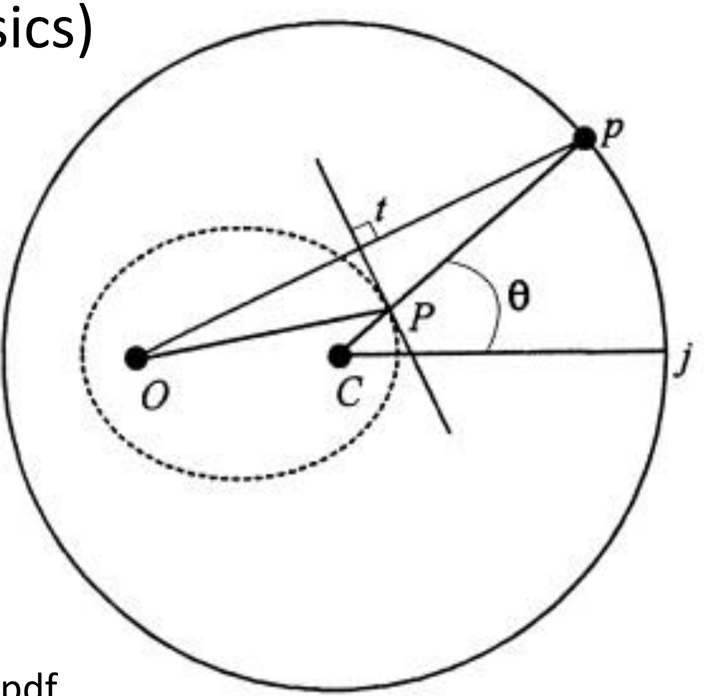
The theme of "Derivation" in different works

- Geometric derivation

  (Feynman Lectures on Physics)



http://utf.mff.cuni.cz/~podolsky/Newton/JEVICK04.pdf

The theme of "Derivation" in different works

- Pure academic knowledge of integral and differential calculus

DERIVATION 2. [Scaling] Suppose that $\vec{r}(t)$ is a solution to Kepler's eqn: $\frac{d^2}{dt^2}\vec{r} = -\mu\vec{r}/r^3$. We look for a scaling law: $\vec{r}_\lambda(t) = \lambda\vec{r}(\lambda^a t)$ and try to solve for the scaling exponent a. By a scaling law we simply mean that this transformation should transform solutions to solutions. Use the chain rule, setting $\tau = \lambda^a t$ so that $\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \lambda^a\frac{d}{d\tau}$. Then $\frac{d^2}{dt^2}\vec{r}_\lambda = (\lambda^a)^2\frac{d^2}{d\tau^2}(\lambda\vec{r}(\tau) = \lambda^{2a+1}\frac{d^2}{d\tau^2}\vec{r}(\tau)$ But, since $\vec{r}$ is a solution we have that $\frac{d^2}{d\tau^2}\vec{r}(\tau) = -\mu\vec{r}(\tau)/r(\tau)^3$ Consequently, we have that $\frac{d^2}{dt^2}\vec{r}_\lambda = -\lambda^{2a+1}\mu\vec{r}(\tau)/r(\tau)^3$. Now we want this to equal $-\mu\vec{r}_\lambda/r_\lambda^3$ which equals $-\frac{1}{\lambda^2}\mu\vec{r}(\tau)/r(\tau)^3$. This yields the condition: $\lambda^{2a+1} = \lambda^{-2}$ which implies $2a+1=-2$ or $a=-3/2$.

The Kepler scaling is $\vec{r}_\lambda(t) = \lambda\vec{r}(\lambda^{-3/2}t)$

The theme of "Derivation" in different works

- Introduction of new auxiliary quantities

$$\begin{aligned} 2a &= r_0 + r_{\max} \\ &= r_0 + \frac{r_0 \cdot (1+e)}{1-e} \\ &= \frac{r_0 \cdot (1-e) + r_0 \cdot (1+e)}{1-e} \\ &= \frac{2r_0}{1-e}. \end{aligned}$$

http://is.muni.cz/th/106257/prif_b/Keplerovy_zakony_a_matematicka_analyza.pdf

The main objective of Mathematical Physics Seminar

- Engagingly interconnect mathematics and physics at Secondary School

  (e.g. the history of Kepler´s and Newton´s laws could be mentioned)

- Deepen students´ mathematical knowledge at Grammar School

- Reveal academic mathematics to students fractionally

  (students are often caught off guard in the first semester at universities)

Inclusion of Mathematical Physics Seminar in the education

- Preferably in the final year at Secondary School, Technical Lyceum, etc., (in CR 18-19 years old students)

- **Explained and discussed subject matter in mathematics**
- Function
- Analytic geometry in plane
- Conics

- **Explained and discussed subject matter in physics**
- Vector and scalar quantity
- Newton's law of force
- Newton's law of gravity

The thematic plan of the seminar

1. Historical introduction

2. Necessary mathematical knowledge

3. The definition of derivation

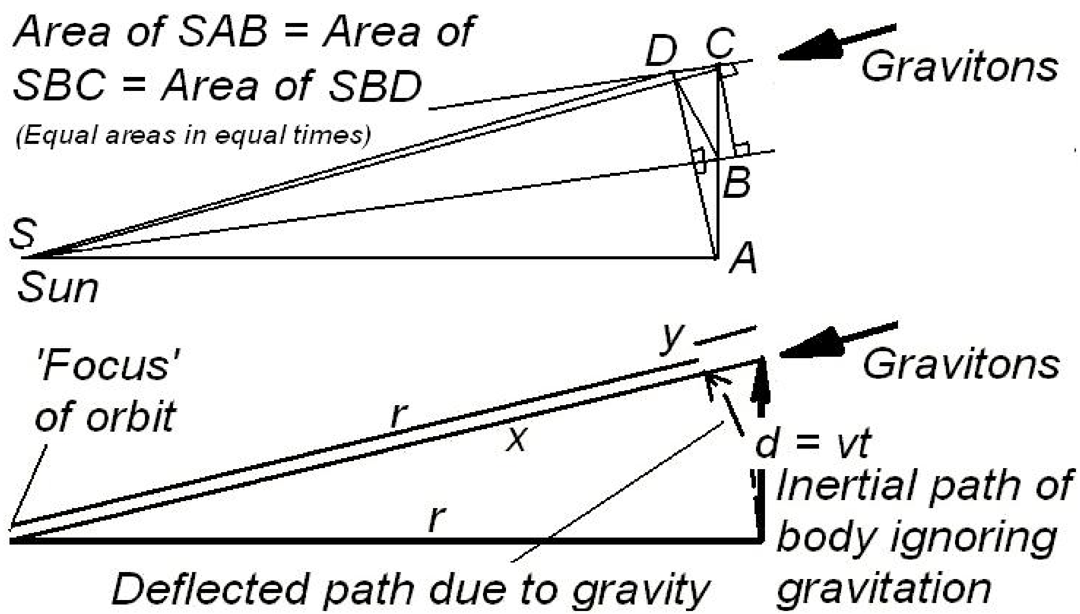
1. Historical introduction

- Explains the historical subtext and methods of Johannes Kepler ´s creative work

- Reminds Newton's laws, used for deriving

Note:

Sir Isaac Newton by demonstrating the consistency between Kepler's laws of planetary motion and his theory of gravitation showed that the motions of objects on Earth and of celestial bodies are governed by the same set of natural laws.



2. Necessary mathematical knowledge

- Revising of basic definitions and mathematical theorems which are subsequently used for deriving

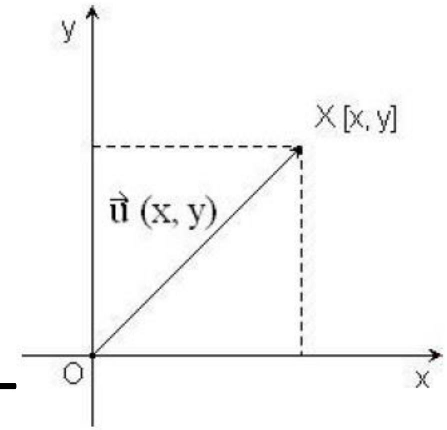
- The following chapters apply:

1. Vectors and operations to them
2. Curve and its physical interpretation
3. Conics (conics equation in polar coordinates)
4. First order differential equations
5. Double integral

1) Vectors and operations to them

Define:

➢ A vector



➢ The vector space over the field T

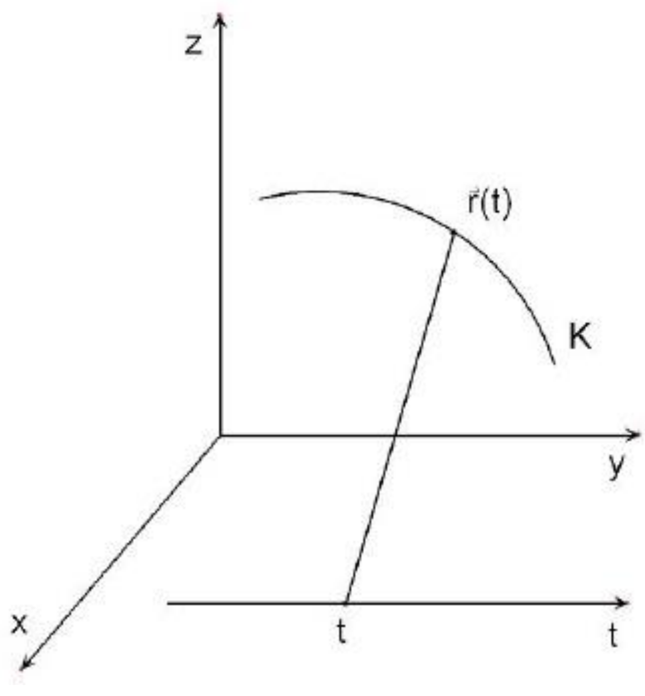
➢ Operations with vectors

- Scalar product $\vec{v}\,.\,\vec{u} = v_1.u_1 + v_2.u_2 + v_3.u_3 + \ldots + v_n.u_n = \sum_{i=1}^{n} v_i.u_i$
- Norm of vectors $|\vec{u}| = \sqrt{\vec{u}.\vec{u}}$
- Vector product

$$\vec{w} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

2) Curves and their interpretations

- There are many introductions of the concept of a curve.
- For our purposes it is sufficient to introduce a concept of a "smooth curve".



Introduction of the following physical concepts:

- Trajectory
- Instantaneous velocity vector

$$\dot{\vec{r}}(t_0) = \lim_{\Delta t \to 0} \frac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t}$$

- Vector of immediate acceleration

$$\ddot{\vec{r}}(t_0)$$

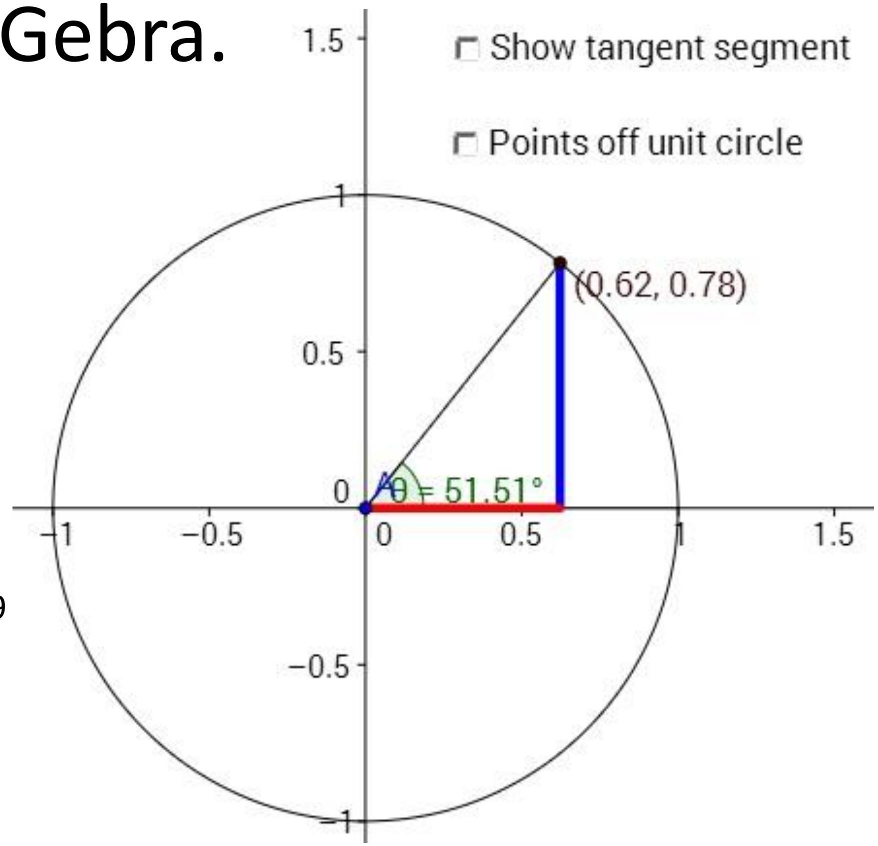
3) Conics

- The knowledge of the equations of conic sections in polar coordinates is required.
- First, introduce polar coordinates.
- Then derive the equations of conic sections in each polar coordinate based on their definitions.

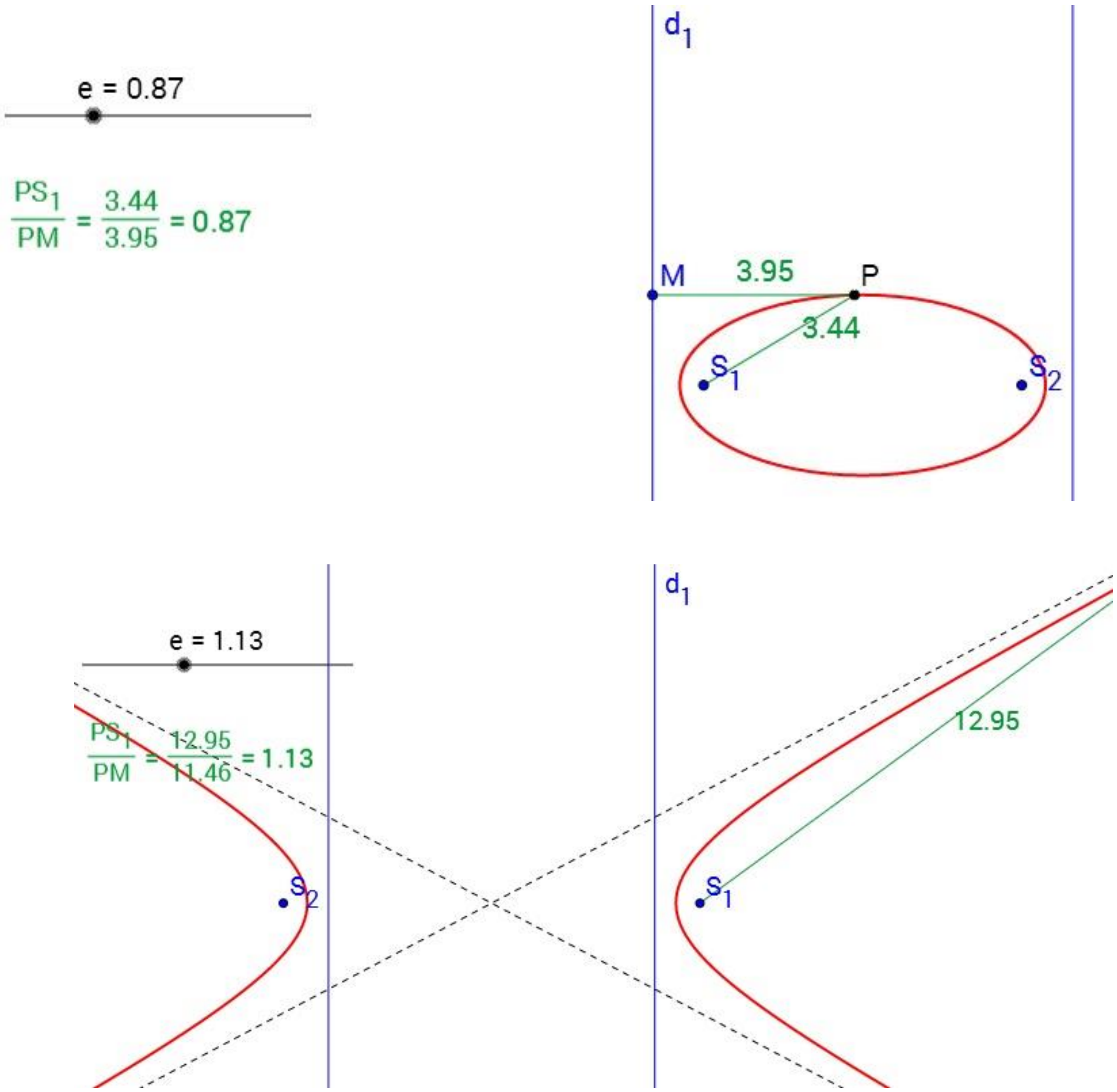
$$\frac{p}{r} = 1 + \varepsilon \cos \varphi,$$

3) Conics

- It is essential to choose an appropriate illustrative example using mathematical software, e. g. GeoGebra.



- http://geogebratube.org/student/m111469



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4) First order differential equations

- In our case it is a simple general differential equation of the first order, which has a shape

$$y' = f(x, y)$$

and it is solved by a method of variables separation

$$y(x) = H^{-1}(G(x) + C)$$

4) Problematic integral of an irrational function

- One of the options is to present it to the students as a formula.

$$I = \int \frac{\mathrm{d}x}{x\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-c}} \arcsin \frac{2c+bx}{x\,\sqrt{b^2-4ac}} + \text{konst.},$$

$$\text{kde } c<0\ ,\ x>0$$

- Excellent students may derive it with the help of the teacher (practicing the method of substitutoin and modifications of expressions).

5) Double integral

- It is only an attempt at a definition that corresponds to the Riemann´s sums.

$$\Omega = \langle a, b \rangle \times \langle c, d \rangle$$

$$\iint_\Omega f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{n \to \infty} \lim_{m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f\left(\tilde{x}_i, \tilde{y}_j\right) \Delta x_i \Delta y_j$$

5) Double integral

- For the proof of Kepler's second law three important theorems on double integrals are used:
  - Fubini´s theorem
  - Substitution theorem in double integrals
  - The level set method

$$|\Omega| = \iint_D abr\,\mathrm{d}r\mathrm{d}\varphi = ab\int_0^1 r\left(\int_0^{2\pi}\mathrm{d}\varphi\right)\mathrm{d}r =$$

$$= ab\int_0^1 2\pi r\,\mathrm{d}r = 2\pi ab\left[\frac{r^2}{2}\right]_0^1 = 2\pi ab\frac{1}{2}$$

$$P = \pi ab$$

3. The Derivation

I have chosen the procedure from the first through the second to the third Kepler's law.

This procedure is chosen while intepreting the derivation at schools.

In other literature sources, however, a different approach is applied. First, from the second to the first, and then to the third law, which is in accordance with history.

The derivation of Kepler's 1st Law

- For the derivation we use two Newton's laws

$$\left.\begin{aligned}\vec{F} &= m\ddot{\vec{r}} \\ \vec{F} &= -\tfrac{k}{r^3}.\vec{r}\end{aligned}\right\} \Rightarrow \ddot{\vec{r}} + \frac{k}{m}\frac{1}{r^3}\vec{r} = 0$$

The derivation of Kepler's 1st Law

- The procedure of the derivation of Kepler's 1st law is as follows:

1. First, we prove that the planetary motion is planar.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vec{r}} \times \vec{r}\right) = \ddot{\vec{r}} \times \vec{r} + \underbrace{\dot{\vec{r}} \times \dot{\vec{r}}}_{=0} = -\frac{k}{m}\frac{1}{r^3}\underbrace{\vec{r} \times \vec{r}}_{=0} = 0$$

Parts of the derivation

2. Subsequently, by using a mathematical aparat, we can determine what type of curve it is.

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ \ddot{x} &= \ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin - r\ddot{\varphi}\sin\varphi + r\dot{\varphi}^2\cos\varphi \\ \ddot{y} &= \ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos + r\ddot{\varphi}\cos\varphi - r\dot{\varphi}^2\sin\varphi \end{aligned}$$

$$\ddot{\vec{r}} + \frac{k}{m}\frac{1}{r^3}\vec{r} = 0$$

$$\ddot{r} - r\dot{\varphi}^2 + \frac{k}{m}\frac{1}{r^2} = 0 \quad \Rightarrow \quad \ddot{r} - \frac{r_0^2 . v_0^2}{r^3} + \frac{k}{m}\frac{1}{r^2} = 0$$

$$2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\varphi}\right) = 0$$

Parts of the derivation

$$\dot{r}^2 + \frac{r_0^2 . v_0^2}{r^2} - \frac{2k}{m}\frac{1}{r} = v_0^2 - \frac{2k}{m}\frac{1}{r_0}$$

The solution of a newly-emerged differential equation is a demanding step. Students could effectively practise the modifications of fractions.

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{v_0^2 - \frac{2k}{m}\frac{1}{r_0} - \frac{r_0^2 . v_0^2}{r^2} + \frac{2k}{m}\frac{1}{r}}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} / \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{r_0 . v_0}{r\sqrt{\left(v_0^2 - \frac{2k}{m}\frac{1}{r_0}\right) r^2 + \frac{2k}{m} r - r_0^2 . v_0^2}}.$$

$$\varphi = r_0 . v_0 \int_{r_0}^{r} \frac{\mathrm{d}\varrho}{\varrho\sqrt{\mathrm{a}\varrho^2 + \mathrm{b}\varrho + \mathrm{c}}}$$

Parts of the derivation

$$= \frac{r_0.v_0}{\sqrt{-\mathrm{c}}}\left[\arcsin\frac{\mathrm{b}\varrho+2\mathrm{c}}{\varrho\sqrt{\mathrm{b}^2-4\mathrm{ac}}}\right]_{r_0}^{r} = \left[\arcsin\frac{\frac{2k}{m}\varrho-2r_0^2.v_0^2}{2\varrho\left|\frac{k}{m}-r_0.v_0^2\right|^2}\right]_{r_0}^{r} =$$

$$= \left[\arcsin\frac{k.\varrho-r_0^2.v_0^2.m}{\varrho\left|k-r_0.v_0^2.m\right|}\right]_{r_0}^{r} = sgn\left(k-r_0v_0^2m\right).\left[\arcsin\frac{k.\varrho-r_0^2.v_0^2.m}{\varrho\left(k-r_0.v_0^2.m\right)}\right]_{r_0}^{r} =$$

$$= sgn\left(k-r_0v_0^2m\right).\left[\arcsin\frac{k.r-r_0^2.v_0^2.m}{r\left(k-r_0.v_0^2.m\right)}-\frac{\pi}{2}\right]$$

Vypočteme si $\cos\varphi$:

$$\cos\varphi = \cos\left\{sgn\left(k-r_0v_0^2m\right).\left[\arcsin\frac{k.r-r_0^2.v_0^2.m}{r\left(k-r_0.v_0^2.m\right)}-\frac{\pi}{2}\right]\right\}$$

Parts of the derivation

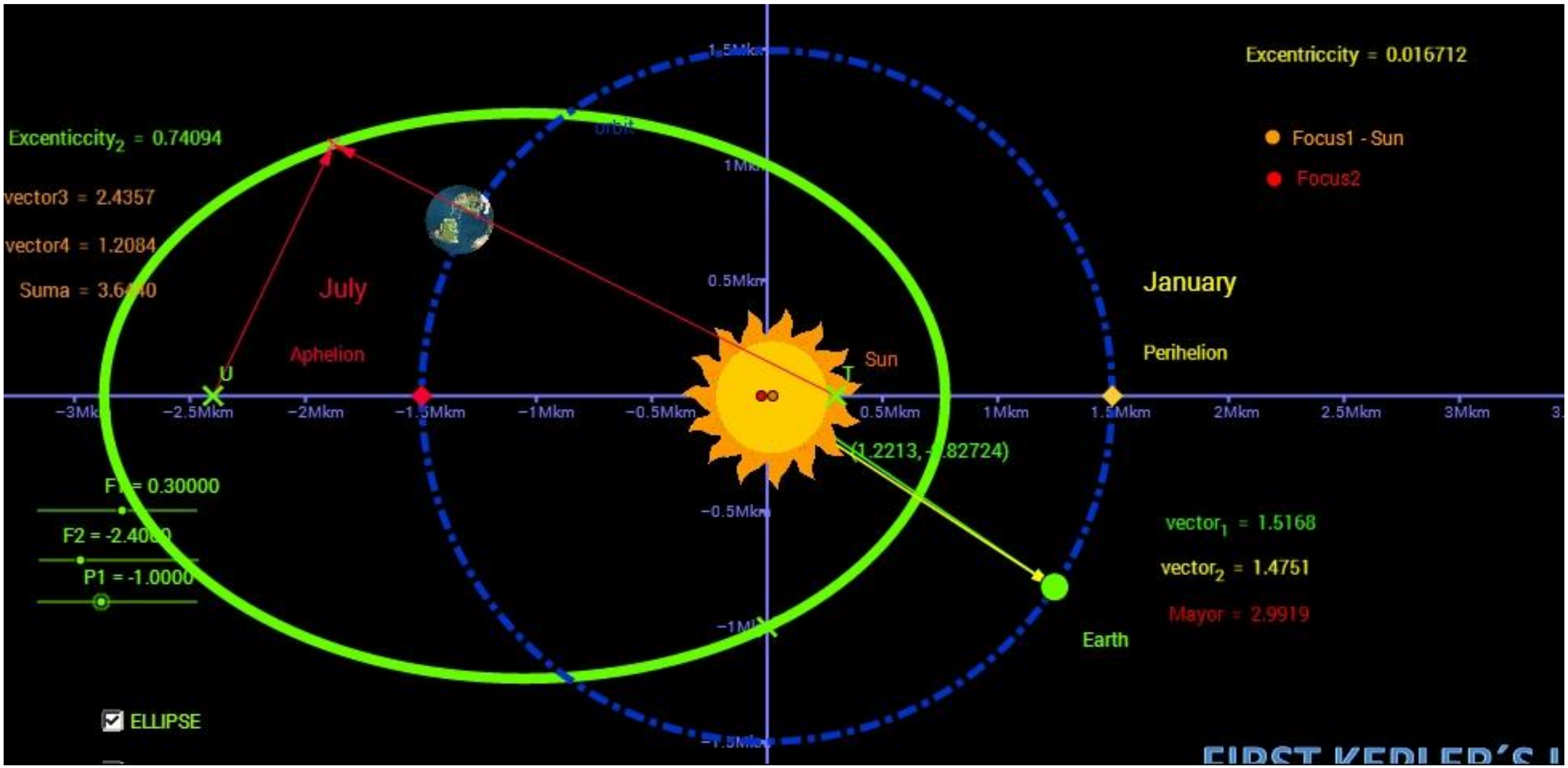
$$\vec{r}(\varphi) = \frac{r_0^2.v_0^2.m}{k - (k - r_0.v_0^2.m)\cos\varphi} = \frac{\frac{m}{k} r_0^2.v_0^2}{1 - \left(1 - \frac{m}{k} r_0.v_0^2\right)\cos\varphi}$$

$$r = \frac{\mathrm{p}}{1 + \varepsilon\cos\varphi} \qquad \mathrm{p} = \frac{m}{k} r_0^2.v_0^2 \geq 0$$

$$\varepsilon = \frac{m}{k} r_0.v_0^2 - 1 \geq -1$$

$$0 < |\varepsilon| < 1$$

Part of the derivation - GeoGebra



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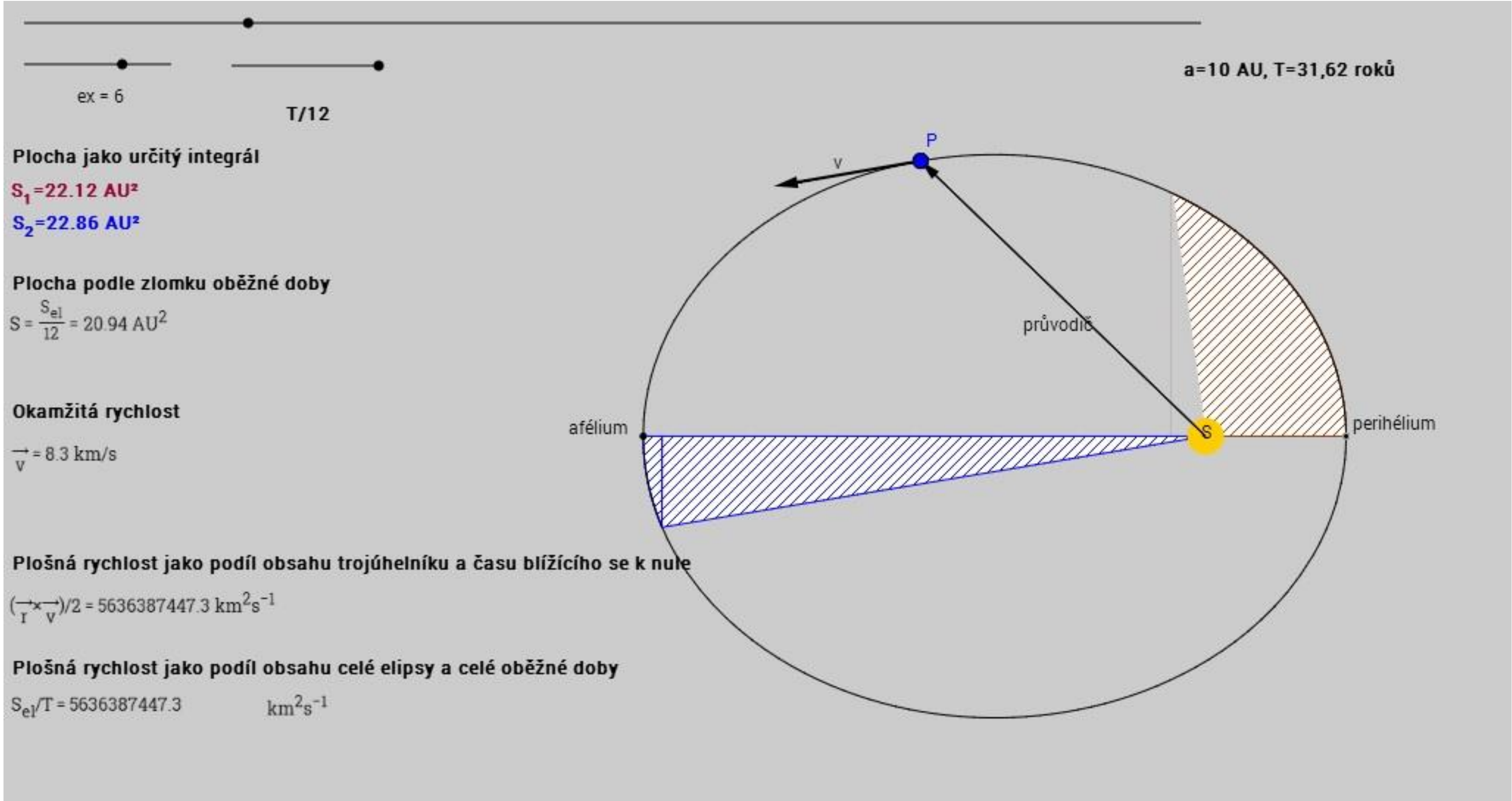
The derivation of Kepler's 2nd Law

- Considering the application of double integral, this derivation is not suitable for secondary schools. It can be replaced by a physical applet or appropriate mathematical software.

Part of the derivation - GeoGebra



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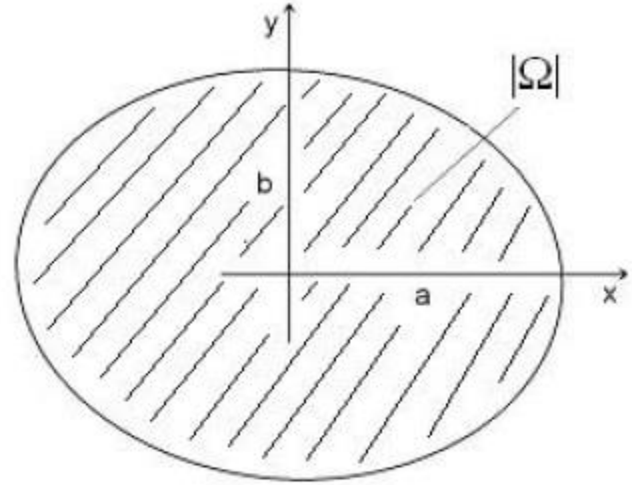
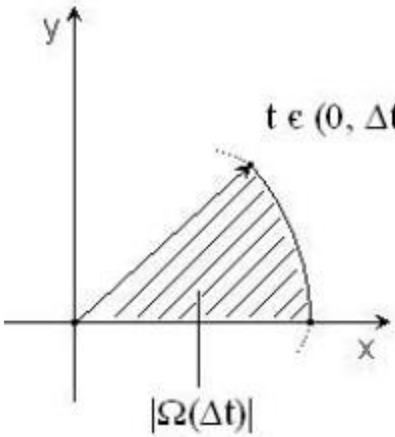
Part of the derivation





$$|\Omega(\Delta t)| = \iint_\Omega \mathrm{d}x\mathrm{d}y$$

$$\int_0^{g(\Delta t)} \left(\int_0^{h(\varphi)} r\mathrm{d}r \right) \mathrm{d}\varphi = \int_0^{g(\Delta t)} \frac{1}{2} h^2(\varphi)\, \mathrm{d}\varphi$$

$$\frac{1}{2} \int_0^{\Delta t} h^2(g(t))\, \dot{g}(t)\, \mathrm{d}t = \frac{1}{2} \int_0^{\Delta t} \underbrace{f^2(t)\, \dot{g}(t)}_{v_0 r_0} \mathrm{d}t = \frac{1}{2} v_0 r_0 \Delta t$$

However, talented students while deriving may link the definition of the definite integral with the area determined by the curve in the plane (the idea of I. Newton).

The derivation of Kepler's 3rd Law

- In the process of derivation of Kepler's first law, we have determined that the shape of the curve, which the planets revolve around the Sun, is an ellipse.

- For simplicity, it can be assumed that the shape of the curve is a circle. I am offering this simplification in my work. In the process, the movement on the ellipse could be tried by more advanced students.

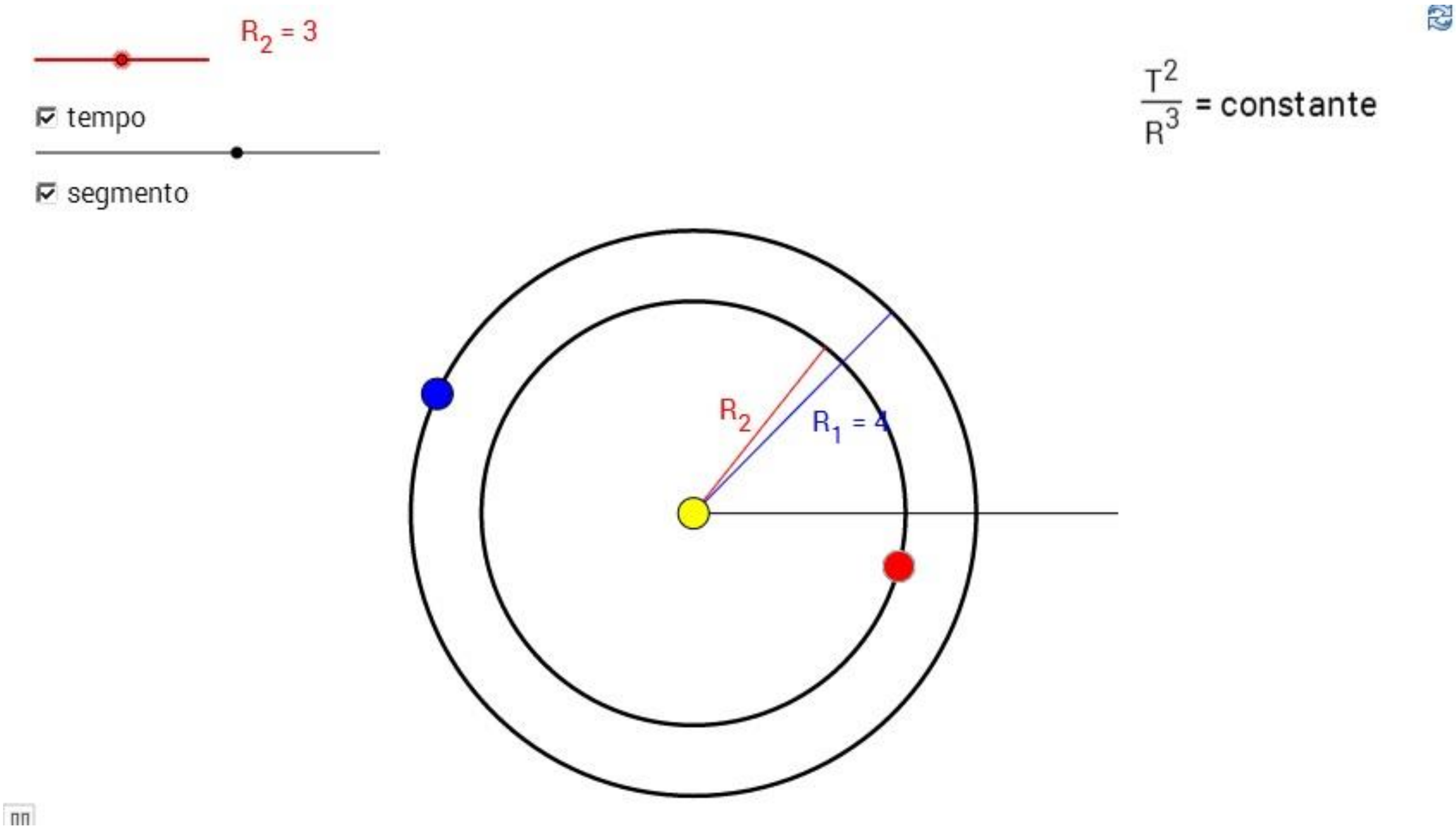
Part of the derivation

$$\sqrt{\frac{\kappa M}{r}} = \frac{2\pi r}{T}$$

$$\frac{r^3}{T^2} = \frac{\kappa M}{4\pi^2}$$

$$\frac{r^3}{T^2} = \text{konst.}$$

Part of the derivation - GeoGebra



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Summary

+ Building interdisciplinary relations
+ The importance of mathematics in Natural Sciences
+ Acquiring and interconnection of physical and mathematical skills

– Demanding mathematical aparat

Thank you for your attention

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