CAS in Teaching Linear Algebra: From Diagnosis, Connection, Deepening to Application

Wen-Haw Chen

Department of Applied Mathematics
Tunghai University
Taichung 40704, TAIWAN.
E-Mail: whchen@thu.edu.tw

Technology in Mathematics Education (TIME 2014)
Krems, Austria.
1^{st} - 5^{th} July, 2014

Abstract

Abstract

- Linear algebra is a fundamental and important subject for students in science, engineering and management schools etc.

Abstract

- Linear algebra is a fundamental and important subject for students in science, engineering and management schools etc.
- Hence the study on the improvement of teaching and learning of linear algebra has been a crucial topic in undergraduate mathematics education.

Abstract

- Linear algebra is a fundamental and important subject for students in science, engineering and management schools etc.
- Hence the study on the improvement of teaching and learning of linear algebra has been a crucial topic in undergraduate mathematics education.
- This is an action research about integrate CAS in a linear algebra course carried out in the academic year 2013-14 at Tunghai University in Taiwan.

Abstract

- Linear algebra is a fundamental and important subject for students in science, engineering and management schools etc.
- Hence the study on the improvement of teaching and learning of linear algebra has been a crucial topic in undergraduate mathematics education.
- This is an action research about integrate CAS in a linear algebra course carried out in the academic year 2013-14 at Tunghai University in Taiwan.
- Our teaching strategies is divided the academic-year course into four stages: diagnosis, connection, deepening and application together with using CAS in linear algebra.

Questions for teachers

Interesting questions for considering linear algebra teachers (Day and Kalman, 1999; Dikovic, 2007):

Questions for teachers

Interesting questions for considering linear algebra teachers (Day and Kalman, 1999; Dikovic, 2007):

- What is optimal to teach at the first course and what should be a student's previous knowledge?

Questions for teachers

Interesting questions for considering linear algebra teachers (Day and Kalman, 1999; Dikovic, 2007):

- What is optimal to teach at the first course and what should be a student's previous knowledge?
- Which degree of abstraction should the teaching aim to?

Questions for teachers

Interesting questions for considering linear algebra teachers (Day and Kalman, 1999; Dikovic, 2007):

- What is optimal to teach at the first course and what should be a student's previous knowledge?
- Which degree of abstraction should the teaching aim to?
- Which facts should be used in the proofs?

Questions for teachers

Interesting questions for considering linear algebra teachers (Day and Kalman, 1999; Dikovic, 2007):

- What is optimal to teach at the first course and what should be a student's previous knowledge?
- Which degree of abstraction should the teaching aim to?
- Which facts should be used in the proofs?
- How much time should be spent on some lesson topics?

Questions for teachers

Interesting questions for considering linear algebra teachers (Day and Kalman, 1999; Dikovic, 2007):

- What is optimal to teach at the first course and what should be a student's previous knowledge?
- Which degree of abstraction should the teaching aim to?
- Which facts should be used in the proofs?
- How much time should be spent on some lesson topics?
- What is the measure of qualitative understanding of basic principles and conceptual learning in linear algebra?

Questions for teachers

Interesting questions for considering linear algebra teachers (Day and Kalman, 1999; Dikovic, 2007):

- What is optimal to teach at the first course and what should be a student's previous knowledge?
- Which degree of abstraction should the teaching aim to?
- Which facts should be used in the proofs?
- How much time should be spent on some lesson topics?
- What is the measure of qualitative understanding of basic principles and conceptual learning in linear algebra?
- Whether, when, and how to use technology?

Goal and Strategies

- Goals (effective teaching method):

Goal and Strategies

- Goals (effective teaching method):

Goal and Strategies

- Goals (effective teaching method):
  1. Connect smoothly a student's previous knowledge to linear algebra course.

Goal and Strategies

- Goals (effective teaching method):
  1. Connect smoothly a student's previous knowledge to linear algebra course.
  2. Enrich traditional lecturing in linear algebra class in order to improve meaningful learning.

Goal and Strategies

- Goals (effective teaching method):
  1. Connect smoothly a student's previous knowledge to linear algebra course.
  2. Enrich traditional lecturing in linear algebra class in order to improve meaningful learning.
  3. Provide to students motivation for learning abstract definitions and theorems with proofs.

Goal and Strategies

- Goals (effective teaching method):
  1. Connect smoothly a student's previous knowledge to linear algebra course.
  2. Enrich traditional lecturing in linear algebra class in order to improve meaningful learning.
  3. Provide to students motivation for learning abstract definitions and theorems with proofs.
  4. Helping students realize the usefulness of linear algebra by applying it to solve various problems.

Goal and Strategies

- Goals (effective teaching method):
  1. Connect smoothly a student's previous knowledge to linear algebra course.
  2. Enrich traditional lecturing in linear algebra class in order to improve meaningful learning.
  3. Provide to students motivation for learning abstract definitions and theorems with proofs.
  4. Helping students realize the usefulness of linear algebra by applying it to solve various problems.
- Strategies (all with the aid of CAS):

Goal and Strategies

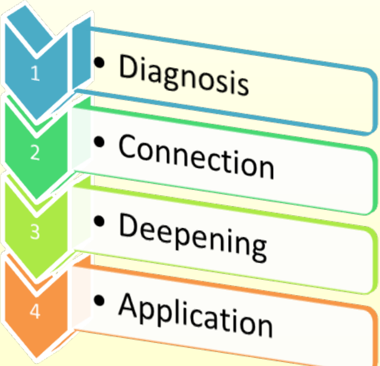
- Goals (effective teaching method):
  1. Connect smoothly a student's previous knowledge to linear algebra course.
  2. Enrich traditional lecturing in linear algebra class in order to improve meaningful learning.
  3. Provide to students motivation for learning abstract definitions and theorems with proofs.
  4. Helping students realize the usefulness of linear algebra by applying it to solve various problems.

- Strategies (all with the aid of CAS):

Goal and Strategies

- Goals (effective teaching method):
  1. Connect smoothly a student's previous knowledge to linear algebra course.
  2. Enrich traditional lecturing in linear algebra class in order to improve meaningful learning.
  3. Provide to students motivation for learning abstract definitions and theorems with proofs.
  4. Helping students realize the usefulness of linear algebra by applying it to solve various problems.

- Strategies (all with the aid of CAS):



Why Involves CAS

Mathematics software packages such as MALAB have the following powerful and numerous functions (Dikovic, 2007):

Why Involves CAS

Mathematics software packages such as MALAB have the following powerful and numerous functions (Dikovic, 2007):

- Instantaneous numerical and symbolic calculations;

Why Involves CAS

Mathematics software packages such as MALAB have the following powerful and numerous functions (Dikovic, 2007):

- Instantaneous numerical and symbolic calculations;
- Data collecting, analysis, exploration, and visualization;

Why Involves CAS

Mathematics software packages such as MALAB have the following powerful and numerous functions (Dikovic, 2007):

- Instantaneous numerical and symbolic calculations;
- Data collecting, analysis, exploration, and visualization;
- Modeling, simulation, and prototyping;

Why Involves CAS

Mathematics software packages such as MALAB have the following powerful and numerous functions (Dikovic, 2007):

- Instantaneous numerical and symbolic calculations;
- Data collecting, analysis, exploration, and visualization;
- Modeling, simulation, and prototyping;
- Presentation graphics and animation in 2D and 3D;

Why Involves CAS

Mathematics software packages such as MALAB have the following powerful and numerous functions (Dikovic, 2007):

- Instantaneous numerical and symbolic calculations;
- Data collecting, analysis, exploration, and visualization;
- Modeling, simulation, and prototyping;
- Presentation graphics and animation in 2D and 3D;
- Application development.

Why Involves CAS

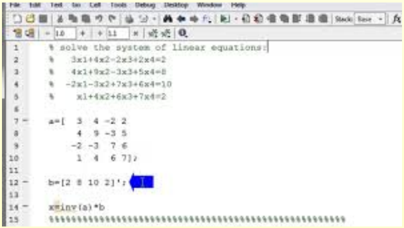
Mathematics software packages such as MALAB have the following powerful and numerous functions (Dikovic, 2007):

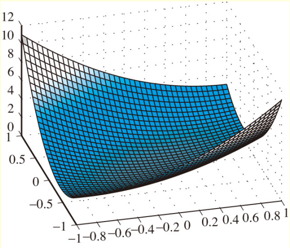
- Instantaneous numerical and symbolic calculations;
- Data collecting, analysis, exploration, and visualization;
- Modeling, simulation, and prototyping;
- Presentation graphics and animation in 2D and 3D;
- Application development.

Why Involves CAS

Mathematics software packages such as MALAB have the following powerful and numerous functions (Dikovic, 2007):

- Instantaneous numerical and symbolic calculations;
- Data collecting, analysis, exploration, and visualization;
- Modeling, simulation, and prototyping;
- Presentation graphics and animation in 2D and 3D;
- Application development.





First Stage: Diagnosis-1

Objectives:

First Stage: Diagnosis-1

Objectives:

- In the first stage *diagnosis*, we assess students' prior knowledge before processing the course.

First Stage: Diagnosis-1

Objectives:

- In the first stage *diagnosis*, we assess students' prior knowledge before processing the course.
- Help students to review the notions learned in senior high school about solve linear systems ,vectors, inner product, matrix and determinate etc.

First Stage: Diagnosis-1

Objectives:

- In the first stage *diagnosis*, we assess students' prior knowledge before processing the course.
- Help students to review the notions learned in senior high school about solve linear systems ,vectors, inner product, matrix and determinate etc.

First Stage: Diagnosis-1

Objectives:

- In the first stage *diagnosis*, we assess students' prior knowledge before processing the course.
- Help students to review the notions learned in senior high school about solve linear systems ,vectors, inner product, matrix and determinate etc.

Approaches:

First Stage: Diagnosis-1

Objectives:

- In the first stage *diagnosis*, we assess students' prior knowledge before processing the course.
- Help students to review the notions learned in senior high school about solve linear systems ,vectors, inner product, matrix and determinate etc.

Approaches:

- Design an in-nominate questionnaire (instead of a test, without computation) and ask students to complete it before the first class.

First Stage: Diagnosis-1

Objectives:

- In the first stage *diagnosis*, we assess students' prior knowledge before processing the course.
- Help students to review the notions learned in senior high school about solve linear systems ,vectors, inner product, matrix and determinate etc.

Approaches:

- Design an in-nominate questionnaire (instead of a test, without computation) and ask students to complete it before the first class.
- Have a test that students can solve the problems with the aid of CAS (basic MALAB or Math Apps in android or apple etc.).

First Stage: Diagnosis-2

A sample of questions in the questionnaire (for 66 students, which are middle level between all 2013 freshmen in Taiwan) :

First Stage: Diagnosis-2

A sample of questions in the questionnaire (for 66 students, which are middle level between all 2013 freshmen in Taiwan) :

- Q: *Can you write down the equations of three non-parallel planes in the space* $\mathbb{R}^3$ *which intersect in a straight line?*

First Stage: Diagnosis-2

A sample of questions in the questionnaire (for 66 students, which are middle level between all 2013 freshmen in Taiwan) :

- Q: *Can you write down the equations of three non-parallel planes in the space* $\mathbb{R}^3$ *which intersect in a straight line?*
  1. Yes, I can. (Please write down an example below).

First Stage: Diagnosis-2

A sample of questions in the questionnaire (for 66 students, which are middle level between all 2013 freshmen in Taiwan) :

- Q: *Can you write down the equations of three non-parallel planes in the space* $\mathbb{R}^3$ *which intersect in a straight line?*
  1. Yes, I can. (Please write down an example below).
  2. I have heard about it but I cannot write an exactly example.

First Stage: Diagnosis-2

A sample of questions in the questionnaire (for 66 students, which are middle level between all 2013 freshmen in Taiwan) :

- Q: *Can you write down the equations of three non-parallel planes in the space* $\mathbb{R}^3$ *which intersect in a straight line?*
  1. Yes, I can. (Please write down an example below).
  2. I have heard about it but I cannot write an exactly example.
  3. No, I have no idea about it.

First Stage: Diagnosis-2

A sample of questions in the questionnaire (for 66 students, which are middle level between all 2013 freshmen in Taiwan) :

- Q: *Can you write down the equations of three non-parallel planes in the space* $\mathbb{R}^3$ *which intersect in a straight line?*
  1. Yes, I can. (Please write down an example below).
  2. I have heard about it but I cannot write an exactly example.
  3. No, I have no idea about it.
- Statistics :

	Yes with examples	Have heard about it	No idea
# of students	10 (correct) ; 4 (wrong)	48	4
percentage	15% ; 6%	73%	6%

Second Stage: Connection-1

Objectives:

Second Stage: Connection-1

Objectives:

- Develop in this stage the material for the bridging course according to the result in the diagnosis stage.

Approaches:

Second Stage: Connection-1

Objectives:

- Develop in this stage the material for the bridging course according to the result in the diagnosis stage.
- Help students to establish their own *Learning Profile*.

Approaches:

Second Stage: Connection-1

Objectives:

- Develop in this stage the material for the bridging course according to the result in the diagnosis stage.
- Help students to establish their own *Learning Profile*.

Approaches:

Second Stage: Connection-1

Objectives:

- Develop in this stage the material for the bridging course according to the result in the diagnosis stage.
- Help students to establish their own *Learning Profile*.

Approaches:

- Analyze the questionnaire and the test in the last stage. Collect those concepts that students are indistinct or impenetrable as the contents for the first week class.

Second Stage: Connection-1

Objectives:

- Develop in this stage the material for the bridging course according to the result in the diagnosis stage.
- Help students to establish their own *Learning Profile*.

Approaches:

- Analyze the questionnaire and the test in the last stage. Collect those concepts that students are indistinct or impenetrable as the contents for the first week class.
- Use MALAB to help teaching.

Second Stage: Connection-1

Objectives:

- Develop in this stage the material for the bridging course according to the result in the diagnosis stage.
- Help students to establish their own *Learning Profile*.

Approaches:

- Analyze the questionnaire and the test in the last stage. Collect those concepts that students are indistinct or impenetrable as the contents for the first week class.
- Use MALAB to help teaching.
- Connect to the related topics in university linear algebra.

Second Stage: Connection-1

Objectives:

- Develop in this stage the material for the bridging course according to the result in the diagnosis stage.
- Help students to establish their own *Learning Profile*.

Approaches:

- Analyze the questionnaire and the test in the last stage. Collect those concepts that students are indistinct or impenetrable as the contents for the first week class.
- Use MALAB to help teaching.
- Connect to the related topics in university linear algebra.
- Students have to record their learning processes and give feedback to the instructor or TA through the online learning moodle.

Second Stage: Connection-2

Fact from diagnosis: About 80% students are not familiar with the notion of the equations of three non-parallel planes in the space $\mathbb{R}^3$ which intersect in a straight line

Second Stage: Connection-2

Fact from diagnosis: About 80% students are not familiar with the notion of the equations of three non-parallel planes in the space $\mathbb{R}^3$ which intersect in a straight line

- Material for remedial teaching:

Second Stage: Connection-2

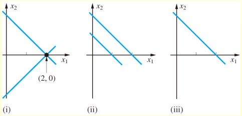
Fact from diagnosis: About 80% students are not familiar with the notion of the equations of three non-parallel planes in the space $\mathbb{R}^3$ which intersect in a straight line

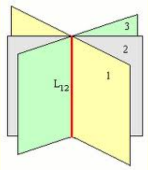
- Material for remedial teaching:

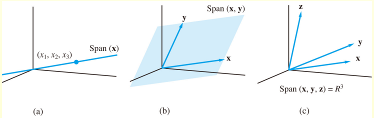
Second Stage: Connection-2

Fact from diagnosis: About 80% students are not familiar with the notion of the equations of three non-parallel planes in the space $\mathbb{R}^3$ which intersect in a straight line

- Material for remedial teaching:



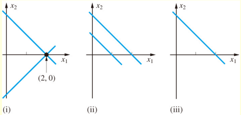


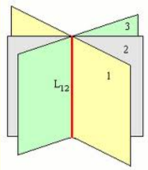


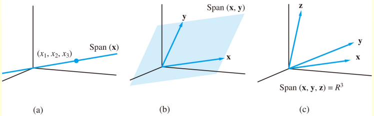
Second Stage: Connection-2

Fact from diagnosis: About 80% students are not familiar with the notion of the equations of three non-parallel planes in the space $\mathbb{R}^3$ which intersect in a straight line

- Material for remedial teaching:







- Connect to the matrix representation of system of linear equations, Gauss elimination and the span of vectors in $\mathbb{R}^3$.

Third Stage: Deepening-1

Objectives:

Third Stage: Deepening-1

Objectives:

- Introduce smoothly to students the abstract notions in linear algebra.

Third Stage: Deepening-1

Objectives:

- Introduce smoothly to students the abstract notions in linear algebra.
- Make students realize *How to Prove*.

Third Stage: Deepening-1

Objectives:

- Introduce smoothly to students the abstract notions in linear algebra.
- Make students realize *How to Prove*.

Third Stage: Deepening-1

Objectives:

- Introduce smoothly to students the abstract notions in linear algebra.
- Make students realize *How to Prove*.

Approaches:

Third Stage: Deepening-1

Objectives:

- Introduce smoothly to students the abstract notions in linear algebra.
- Make students realize *How to Prove*.

Approaches:

- By use of the *Problem-Based Learning* (PBL) Model. Each topic is introduced by a problem and then give some hints. Students have to learn the notions through answer a series questions.

Third Stage: Deepening-1

Objectives:

- Introduce smoothly to students the abstract notions in linear algebra.
- Make students realize *How to Prove*.

Approaches:

- By use of the *Problem-Based Learning* (PBL) Model. Each topic is introduced by a problem and then give some hints. Students have to learn the notions through answer a series questions.
- Students become stakeholders and teacher as a trainer in cognition and meta-cognition.

Third Stage: Deepening-1

Objectives:

- Introduce smoothly to students the abstract notions in linear algebra.
- Make students realize *How to Prove*.

Approaches:

- By use of the *Problem-Based Learning* (PBL) Model. Each topic is introduced by a problem and then give some hints. Students have to learn the notions through answer a series questions.
- Students become stakeholders and teacher as a trainer in cognition and meta-cognition.
- Encouraging group cooperation and learning.

Third Stage: Deepening-1

Objectives:

- Introduce smoothly to students the abstract notions in linear algebra.
- Make students realize *How to Prove*.

Approaches:

- By use of the *Problem-Based Learning* (PBL) Model. Each topic is introduced by a problem and then give some hints. Students have to learn the notions through answer a series questions.
- Students become stakeholders and teacher as a trainer in cognition and meta-cognition.
- Encouraging group cooperation and learning.
- Then teacher introduces the abstract proof and extensions.

Third Stage: Deepening-2

Topic: Definitions of Eigenvalues and Eigenvectors

Third Stage: Deepening-2

Topic: Definitions of Eigenvalues and Eigenvectors

- Question: Given an $n \times n$ matrix A. How can you compute A^k for some $k \in \mathbb{N}$ or for $k \to \infty$? (It has many applications such as Markov chain.)

Third Stage: Deepening-2

Topic: Definitions of Eigenvalues and Eigenvectors

- Question: Given an $n \times n$ matrix A. How can you compute A^k for some $k \in \mathbb{N}$ or for $k \to \infty$? (It has many applications such as Markov chain.)
- Hint: How about if A is *similar* to a real diagonal matrix $D = diag[\lambda_1, \ldots, \lambda_n]$ (i.e. $\exists$ a nonsingular matrix X such that $X^{-1}AX = D$)?

Third Stage: Deepening-2

Topic: Definitions of Eigenvalues and Eigenvectors

- Question: Given an $n \times n$ matrix A. How can you compute A^k for some $k \in \mathbb{N}$ or for $k \to \infty$? (It has many applications such as Markov chain.)
- Hint: How about if A is *similar* to a real diagonal matrix $D = diag[\lambda_1, \ldots, \lambda_n]$ (i.e. $\exists$ a nonsingular matrix X such that $X^{-1}AX = D$)?
- Thinking: What are the conditions of making A similar to D?

Third Stage: Deepening-2

Topic: Definitions of Eigenvalues and Eigenvectors

- Question: Given an $n \times n$ matrix A. How can you compute A^k for some $k \in \mathbb{N}$ or for $k \to \infty$? (It has many applications such as Markov chain.)
- Hint: How about if A is *similar* to a real diagonal matrix $D = diag[\lambda_1, \ldots, \lambda_n]$ (i.e. $\exists$ a nonsingular matrix X such that $X^{-1}AX = D$)?
- Thinking: What are the conditions of making A similar to D?
- Hint: What can you say about the null space $N(A - \lambda_i I_n)$ for some $i \in \{1, 2, \ldots, n\}$ and I_n denotes the identity $n \times n$ matrix?

Third Stage: Deepening-2

Topic: Definitions of Eigenvalues and Eigenvectors

- Question: Given an $n \times n$ matrix A. How can you compute A^k for some $k \in \mathbb{N}$ or for $k \to \infty$? (It has many applications such as Markov chain.)
- Hint: How about if A is *similar* to a real diagonal matrix $D = diag[\lambda_1, \ldots, \lambda_n]$ (i.e. $\exists$ a nonsingular matrix X such that $X^{-1}AX = D$)?
- Thinking: What are the conditions of making A similar to D?
- Hint: What can you say about the null space $N(A - \lambda_i I_n)$ for some $i \in \{1, 2, \ldots, n\}$ and I_n denotes the identity $n \times n$ matrix?
- Then the teacher can introduce the definitions of eigenvalues, eigenvectors and diagonalization for an $n \times n$ matrix.

Third Stage: Deepening-2

Topic: Definitions of Eigenvalues and Eigenvectors

- Question: Given an $n \times n$ matrix A. How can you compute A^k for some $k \in \mathbb{N}$ or for $k \to \infty$? (It has many applications such as Markov chain.)
- Hint: How about if A is *similar* to a real diagonal matrix $D = diag[\lambda_1, \ldots, \lambda_n]$ (i.e. $\exists$ a nonsingular matrix X such that $X^{-1}AX = D$)?
- Thinking: What are the conditions of making A similar to D?
- Hint: What can you say about the null space $N(A - \lambda_i I_n)$ for some $i \in \{1, 2, \ldots, n\}$ and I_n denotes the identity $n \times n$ matrix?
- Then the teacher can introduce the definitions of eigenvalues, eigenvectors and diagonalization for an $n \times n$ matrix.
- Extended question: Can you give an example of non-diagonalizable? What are the sufficient and necessary conditions of diagonalizability?

Fourth Stage: Application-1

Objectives:

Fourth Stage: Application-1

Objectives:

- Students can apply learned knowledge to solve the practical problems designed by the characteristics of different fields.

Approaches:

Fourth Stage: Application-1

Objectives:

- Students can apply learned knowledge to solve the practical problems designed by the characteristics of different fields.
- Preparation for in-depth notions.

Approaches:

Fourth Stage: Application-1

Objectives:

- Students can apply learned knowledge to solve the practical problems designed by the characteristics of different fields.
- Preparation for in-depth notions.

Approaches:

Fourth Stage: Application-1

Objectives:

- Students can apply learned knowledge to solve the practical problems designed by the characteristics of different fields.
- Preparation for in-depth notions.

Approaches:

- Checking first students priori knowledge before giving practical problems.

Fourth Stage: Application-1

Objectives:

- Students can apply learned knowledge to solve the practical problems designed by the characteristics of different fields.
- Preparation for in-depth notions.

Approaches:

- Checking first students priori knowledge before giving practical problems.
- Choose the applications which are related to students' learning experience.

Fourth Stage: Application-1

Objectives:

- Students can apply learned knowledge to solve the practical problems designed by the characteristics of different fields.
- Preparation for in-depth notions.

Approaches:

- Checking first students priori knowledge before giving practical problems.
- Choose the applications which are related to students' learning experience.
- For example, the *least square problem* (find the best fitting straight line for giving data) is related to the statistic course together with the use of CAS.

Fourth Stage: Application-1

Objectives:

- Students can apply learned knowledge to solve the practical problems designed by the characteristics of different fields.
- Preparation for in-depth notions.

Approaches:

- Checking first students priori knowledge before giving practical problems.
- Choose the applications which are related to students' learning experience.
- For example, the *least square problem* (find the best fitting straight line for giving data) is related to the statistic course together with the use of CAS.

Fourth Stage: Application-1

Objectives:

- Students can apply learned knowledge to solve the practical problems designed by the characteristics of different fields.
- Preparation for in-depth notions.

Approaches:

- Checking first students priori knowledge before giving practical problems.
- Choose the applications which are related to students' learning experience.
- For example, the *least square problem* (find the best fitting straight line for giving data) is related to the statistic course together with the use of CAS.

Q:*Find the best line* $y = c_0 + c_1 x$ *to fit the three points* $(0,1)$, $(3,4)$ *and* $(6,5)$.

Fourth Stage: Application-1

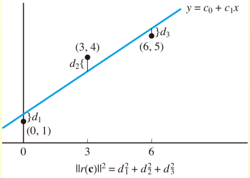
Objectives:

- Students can apply learned knowledge to solve the practical problems designed by the characteristics of different fields.
- Preparation for in-depth notions.

Approaches:

- Checking first students priori knowledge before giving practical problems.
- Choose the applications which are related to students' learning experience.
- For example, the *least square problem* (find the best fitting straight line for giving data) is related to the statistic course together with the use of CAS.

Q:*Find the best line* $y = c_0 + c_1 x$ *to fit the three points* $(0,1)$, $(3,4)$ *and* $(6,5)$.



Fourth Stage: Application-2

In-depth notion: *Geometric Mean of Positively Definite Matrices*

Fourth Stage: Application-2

In-depth notion: *Geometric Mean of Positively Definite Matrices*

- Priori knowledge: definition and elementary properties of P.D. matrices.

$$\underbrace{\begin{pmatrix} a_{11} & x & x & x \\ x & a_{22} & x & x \\ x & x & a_{33} & x \\ x & x & x & a_{44} \end{pmatrix}}_{A} \xrightarrow{1} \underbrace{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & x & a_{33}^{(1)} & x \\ 0 & x & x & a_{44}^{(1)} \end{pmatrix}}_{A^{(1)}} \xrightarrow{2} \underbrace{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & 0 & a_{33}^{(2)} & x \\ 0 & 0 & x & a_{44}^{(2)} \end{pmatrix}}_{A^{(2)}} \xrightarrow{3} \underbrace{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & 0 & a_{33}^{(2)} & x \\ 0 & 0 & 0 & a_{44}^{(3)} \end{pmatrix}}_{A^{(3)} = U}$$

Fourth Stage: Application-2

In-depth notion: *Geometric Mean of Positively Definite Matrices*

- Priori knowledge: definition and elementary properties of P.D. matrices.

$$\underset{A}{\begin{pmatrix} a_{11} & x & x & x \\ x & a_{22} & x & x \\ x & x & a_{33} & x \\ x & x & x & a_{44} \end{pmatrix}} \xrightarrow{1} \underset{A^{(1)}}{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & x & a_{33}^{(1)} & x \\ 0 & x & x & a_{44}^{(1)} \end{pmatrix}} \xrightarrow{2} \underset{A^{(2)}}{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & 0 & a_{33}^{(2)} & x \\ 0 & 0 & x & a_{44}^{(2)} \end{pmatrix}} \xrightarrow{3} \underset{A^{(3)} = U}{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & 0 & a_{33}^{(2)} & x \\ 0 & 0 & 0 & a_{44}^{(3)} \end{pmatrix}}$$

- Thinking: How similar are P.D. matrices and positive numbers?

Fourth Stage: Application-2

In-depth notion: *Geometric Mean of Positively Definite Matrices*

- Priori knowledge: definition and elementary properties of P.D. matrices.

$$\underset{A}{\begin{pmatrix} a_{11} & x & x & x \\ x & a_{22} & x & x \\ x & x & a_{33} & x \\ x & x & x & a_{44} \end{pmatrix}} \xrightarrow{1} \underset{A^{(1)}}{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & x & a_{33}^{(1)} & x \\ 0 & x & x & a_{44}^{(1)} \end{pmatrix}} \xrightarrow{2} \underset{A^{(2)}}{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & 0 & a_{33}^{(2)} & x \\ 0 & 0 & x & a_{44}^{(2)} \end{pmatrix}} \xrightarrow{3} \underset{A^{(3)} = U}{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & 0 & a_{33}^{(2)} & x \\ 0 & 0 & 0 & a_{44}^{(3)} \end{pmatrix}}$$

- Thinking: How similar are P.D. matrices and positive numbers?
- Note $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ are P.D. but $AB = \begin{bmatrix} -1 & 3 \\ -3 & 8 \end{bmatrix}$ isn't.

Fourth Stage: Application-2

In-depth notion: *Geometric Mean of Positively Definite Matrices*

- Priori knowledge: definition and elementary properties of P.D. matrices.

$$\underset{A}{\begin{pmatrix} a_{11} & x & x & x \\ x & a_{22} & x & x \\ x & x & a_{33} & x \\ x & x & x & a_{44} \end{pmatrix}} \xrightarrow{1} \underset{A^{(1)}}{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & x & a_{33}^{(1)} & x \\ 0 & x & x & a_{44}^{(1)} \end{pmatrix}} \xrightarrow{2} \underset{A^{(2)}}{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & 0 & a_{33}^{(2)} & x \\ 0 & 0 & x & a_{44}^{(2)} \end{pmatrix}} \xrightarrow{3} \underset{A^{(3)} = U}{\begin{pmatrix} a_{11} & x & x & x \\ 0 & a_{22}^{(1)} & x & x \\ 0 & 0 & a_{33}^{(2)} & x \\ 0 & 0 & 0 & a_{44}^{(3)} \end{pmatrix}}$$

- Thinking: How similar are P.D. matrices and positive numbers?
- Note $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ are P.D. but $AB = \begin{bmatrix} -1 & 3 \\ -3 & 8 \end{bmatrix}$ isn't.
- By surveying the literatures, we have (Putz & Woronowicz, 1975): *The geometric mean of two positive definite matrices A and B is defined by*

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

Fourth Stage: Application-3

Project: The harmonic-geometric-arithmetic mean inequality holds for positively definite matrices. That is,

$$(\frac{A^{-1}+B^{-1}}{2})^{-1} \leq A\#B \leq (\frac{A+B}{2}).$$

Fourth Stage: Application-3

Project: The harmonic-geometric-arithmetic mean inequality holds for positively definite matrices. That is,

$$(\frac{A^{-1}+B^{-1}}{2})^{-1} \leq A\#B \leq (\frac{A+B}{2}).$$

1. If A, B are commutative, then $A\#B = A^{1/2}B^{1/2} = (AB)^{1/2}$.

Fourth Stage: Application-3

Project: The harmonic-geometric-arithmetic mean inequality holds for positively definite matrices. That is,

$$(\frac{A^{-1}+B^{-1}}{2})^{-1} \leq A\#B \leq (\frac{A+B}{2}).$$

1. If A, B are commutative, then $A\#B = A^{1/2}B^{1/2} = (AB)^{1/2}$.
2. $A\#B = B\#A$.

Fourth Stage: Application-3

Project: The harmonic-geometric-arithmetic mean inequality holds for positively definite matrices. That is,

$$(\frac{A^{-1}+B^{-1}}{2})^{-1} \leq A\#B \leq (\frac{A+B}{2}).$$

1. If A, B are commutative, then $A\#B = A^{1/2}B^{1/2} = (AB)^{1/2}$.
2. $A\#B = B\#A$.
3. Assume that $A_1 \leq A_2$, $B_1 \leq B_2$, then $A_1\#B_1 \leq A_2\#B_2$.

Fourth Stage: Application-3

Project: The harmonic-geometric-arithmetic mean inequality holds for positively definite matrices. That is,

$$(\frac{A^{-1}+B^{-1}}{2})^{-1} \leq A\#B \leq (\frac{A+B}{2}).$$

1. If A, B are commutative, then $A\#B = A^{1/2}B^{1/2} = (AB)^{1/2}$.
2. $A\#B = B\#A$.
3. Assume that $A_1 \leq A_2$, $B_1 \leq B_2$, then $A_1\#B_1 \leq A_2\#B_2$.
4. $(A_1 + A_2)\#(B_1 + B_2) \geq A_1\#B_1 + A_2\#B_2$.

Fourth Stage: Application-3

Project: The harmonic-geometric-arithmetic mean inequality holds for positively definite matrices. That is,

$$(\frac{A^{-1}+B^{-1}}{2})^{-1} \leq A\#B \leq (\frac{A+B}{2}).$$

1. If A, B are commutative, then $A\#B = A^{1/2}B^{1/2} = (AB)^{1/2}$.
2. $A\#B = B\#A$.
3. Assume that $A_1 \leq A_2$, $B_1 \leq B_2$, then $A_1\#B_1 \leq A_2\#B_2$.
4. $(A_1+A_2)\#(B_1+B_2) \geq A_1\#B_1 + A_2\#B_2$.
5. $(A\#B)^{-1} = A^{-1}\#B^{-1}$.

Conclusion and Discussion-1

Measure students' learning

Conclusion and Discussion-1

Measure students' learning

1. 72% of students can state completely the process of how to solve a system of linear equations with matrix representation and give an correct example of 3×3 case with its geometry.

Conclusion and Discussion-1

Measure students' learning

1. 72% of students can state completely the process of how to solve a system of linear equations with matrix representation and give an correct example of 3×3 case with its geometry.
2. 56% of students can write down and justify the conditions for an $n \times n$ diagonalizable matrix, and can give an 3×3 non-diagonalizable matrix.

Conclusion and Discussion-1

Measure students' learning

1. 72% of students can state completely the process of how to solve a system of linear equations with matrix representation and give an correct example of 3×3 case with its geometry.
2. 56% of students can write down and justify the conditions for an $n \times n$ diagonalizable matrix, and can give an 3×3 non-diagonalizable matrix.
3. 85% of students can solve a practical problem about best fitting lines (least square problem).

Conclusion and Discussion-1

Measure students' learning

1. 72% of students can state completely the process of how to solve a system of linear equations with matrix representation and give an correct example of 3×3 case with its geometry.
2. 56% of students can write down and justify the conditions for an $n \times n$ diagonalizable matrix, and can give an 3×3 non-diagonalizable matrix.
3. 85% of students can solve a practical problem about best fitting lines (least square problem).
4. 3 teams of students finished the project of studying the geometric mean of positively definite matrices.

Conclusion and Discussion-1

Measure students' learning

1. 72% of students can state completely the process of how to solve a system of linear equations with matrix representation and give an correct example of 3×3 case with its geometry.
2. 56% of students can write down and justify the conditions for an $n \times n$ diagonalizable matrix, and can give an 3×3 non-diagonalizable matrix.
3. 85% of students can solve a practical problem about best fitting lines (least square problem).
4. 3 teams of students finished the project of studying the geometric mean of positively definite matrices.
5. Students' opinions on difficulties in learning linear algebra :

Conclusion and Discussion-1

Measure students' learning

1. 72% of students can state completely the process of how to solve a system of linear equations with matrix representation and give an correct example of 3×3 case with its geometry.
2. 56% of students can write down and justify the conditions for an $n \times n$ diagonalizable matrix, and can give an 3×3 non-diagonalizable matrix.
3. 85% of students can solve a practical problem about best fitting lines (least square problem).
4. 3 teams of students finished the project of studying the geometric mean of positively definite matrices.
5. Students' opinions on difficulties in learning linear algebra :
    - Use of English textbook (49%); How to write a precise proof (64%).

Conclusion and Discussion-2

Conclusion and Discussion-2

- A linear algebra teacher should understand better students' previous knowledge, how students learn, how to introduce abstract notions, proofs and applications in class, and how to use technology.

Conclusion and Discussion-2

- A linear algebra teacher should understand better students' previous knowledge, how students learn, how to introduce abstract notions, proofs and applications in class, and how to use technology.
- In this action research, we adopt the teaching strategies of four stages: diagnosis, connection, deepening and application together with using CAS to improve the teaching and learning of linear algebra.

Conclusion and Discussion-2

- A linear algebra teacher should understand better students' previous knowledge, how students learn, how to introduce abstract notions, proofs and applications in class, and how to use technology.
- In this action research, we adopt the teaching strategies of four stages: diagnosis, connection, deepening and application together with using CAS to improve the teaching and learning of linear algebra.
- In general, the strategies promote successfully students' motivation and learning effect.

Conclusion and Discussion-2

- A linear algebra teacher should understand better students' previous knowledge, how students learn, how to introduce abstract notions, proofs and applications in class, and how to use technology.
- In this action research, we adopt the teaching strategies of four stages: diagnosis, connection, deepening and application together with using CAS to improve the teaching and learning of linear algebra.
- In general, the strategies promote successfully students' motivation and learning effect.
- Of course, there is no teaching method giving a certain solution to overcome all the difficulties in teaching and learning algebra.

Thank You For Attention!!

Figure: Luce Chapel, Tunghai University