Study of historical geometric problems by means of CAS and DGS

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Abstract. Common use of the computer algebra system wxMaxima and the dynamic mathematics software GeoGebra to solve a geometric problem on conics and locus from an 18th century textbook will be presented. In particular example will be shown of how the use of these programs helped the authors to understand the method that our predecessors used to deal with the conic sections together with the solving the problems. The combination of DGS and CAS has proved its worth to the authors in solving of such problems. The use of the computer changed the old problems, which to our students were originally solved in a rather strange way, into attractive modern problems.

Keywords: Geometry, locus, conics, GeoGebra, wxMaxima.

1 Introduction

Problems that are presented in this paper are selected from the book *Exercitationes Geometricae* by Ioannis Holfeld (I. H. in the following) [1], published by the Jesuit College of St. Clement in Prague in 1773. The exploration of all 47 exercises and their solutions presented in this book provide a reader with an inspiring insight into the methods that were used to study conics before the analytical method was fully established.

The paper shows how the dynamic geometry software GeoGebra can help the reader to learn more about the methods that are used in the book and also to reveal the incompleteness of the original solutions of selected problems. Such a finding becomes an inspiration for their resolution, applying both the original synthetic method and the contemporary analytical method.

2 Exercitationes Geometricae

The latin book *Exercitationes Geometricae* of 65 pages contains 47 solved exercises that are illustrated by 36 figures (which are separated in attachment to the book). The exercises are divided into four parts, each devoted to a specific topic.

The first part includes 9 problems related to the Apollonius definition of parabola. The central notions of these problems are *diameter*, *abscissa* and *ordinate* (more exactly *semi-ordinate*). As a diameter of a parabola we understand any straight line parallel to the parabola's axis, see the line CT in Fig. 1. Then, in Fig. 1 the line segments TF and CT are semi-ordinate and abscissa, respectively, of the point F belonging to the diameter CT. Then a parabola is defined as a curve for which the relation between the semi-ordinate TF and the corresponding abscissa CT of F is given by $TF^2 = p \cdot CT$. The value of the parameter p for a particular diameter is constant and is equal to one quarter of the distance of the intersection C of the diameter and parabola from the directrix. In solutions of the exercises I. H. usually introduces the variables x, y (abscissa and semi-ordinate) relating to the conjugate diameters and he derives equations of conics in these variables. Nowhere, however, does he mention the notion of a coordinate system.

As an example of a problem from the first part of the book let us present problem number 4 (*Problema 4*, [1], p. 6): *You are given two points A and F of a parabola (see Fig. 1, left), the diameter CH and the value of its parameter. Describe the parabola.* The essence of the solution of this task is to determine the directrix and the focus of the parabola from the given elements.

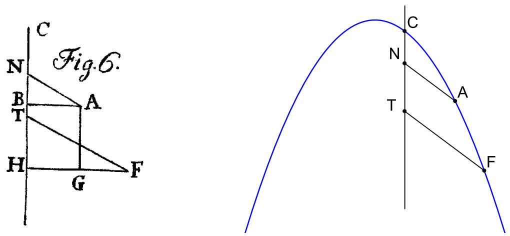

Figure 1: Illustration of the assignment of problem 4 (left) along with an example of a respective parabola (right)

The second part of the book includes 13 solved problems on conics, mainly on parabola. For example, problem number 10 (*Problema 10*, [1], p. 12) is stated as follows: *A secant line RM to a parabola is given (see Fig. 2). Determine the semi-ordinate DN with respect to the diameter OG so that the segment EN bounded by the secant and the corresponding arc of the parabola is the longest of all such segments.*

The third part of the book, from which the example presented in the next section of this paper is selected, brings 19 solved problems on loci. The fourth and last part of the book introduces 6 problems on the computation of the volumes and surfaces of solids of revolution that are obtained by rotating parts of conics around secant or other lines.

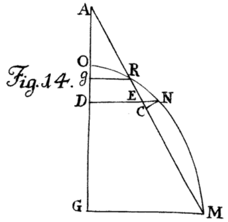

Figure 2: Illustration of the assignment of problem 10

3 Problem 24

This section is devoted to the detailed introduction of one locus problem selected from the book *Exercitationes Geometricae*. Specifically we will deal with problem 24 (*Problema 24*).

Given two lines AM, MB (see Fig. 3); select any point O on the line MB, construct the point R on the line AM so that AR is equal to MO, asking for the set of all intersections of the two lines RB, AO. ([1], p. 27, *Problema 24*)

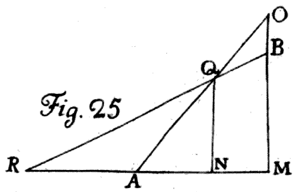

Figure 3: Illustration of the assignment of problem 24

I. H. uses coordinates x, y, but he does not demand the perpendicularity of their axis. His axis have directions of abscissa and ordinate with respect to the chosen diameter (i.e. conjugate directions), which generally differs from the axis of symmetry of the parabola. All despite the fact that the original illustrations show mostly right angles (see Fig. 3). They are often used to illustrate multiple problems and have a largely schematic role. The angle between lines AM and MB can therefore be arbitrary. I. H. begins his solution by labeling lengths of selected segments; $AN = x$, $NQ = y$, $AM = a$, $MB = b$. Then, by the gradual application of the similarity of triangles ANQ and AMO, the equality of segments RA and MO, the similarity of triangles RNQ and RMB and finally by completing the square, I. H. derives the equation of the locus

$$\left(y+\frac{x-b}{2}\right)^2=\frac{(x-b)^2}{4}+\frac{bx^2}{a}, \tag{1}$$

that corresponds to the hyperbola. Its plot for the particular values of a and b in orthogonal coordinates is shown in Fig. 4, left. On the right in the same figure the result can be seen of the use of the software GeoGebra (www.geogebra.org), specifically its tool "Locus". Where, in contrast to I. H. we considered both the positive and the negative $y-$coordinates of point O and also any of the possible positions of point R with respect to point A, see points R, R' in Fig. 4, right. The locus curve plotted by

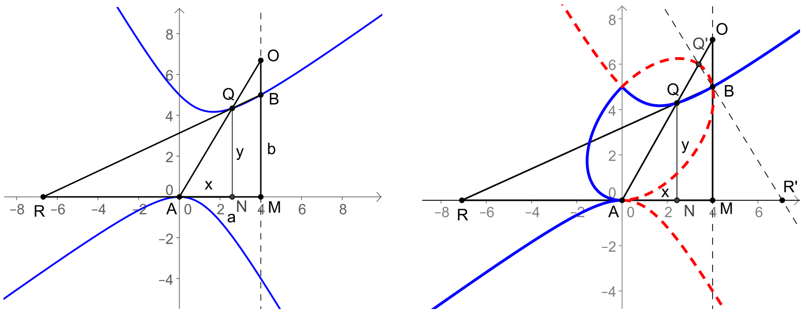

Figure 4: Solution of problem 24 according to I. H. (left) and the solution by means of GeoGebra's tool "Locus" (right)

GeoGebra is a union of the hyperbola, which was identified by I. H. as the solution to problem 24, with another curve, probably an ellipse. Then, the part of the locus drawn by the blue solid line corresponds to the position of R to the left of point A (see R), while the part drawn by the red dashed line corresponds to the position of R to the right of point A (see R').

Why did I. H. not find the ellipse as a part of the solution to problem 24? Its equation can be derived by means of the same procedure in which he found the hyperbola, by taking into consideration the different possible positions of points O and R. It appears that taking into account all possible configurations of given elements was not an obvious part of the solution to the geometrical problem. (These tasks were probably considered as positional tasks.) Their solvers did not usually go outside the boundaries given by a figure. How would we solve problem 24 today, using the computer? Initially, we would probably use GeoGebra to explore the task. Then we would create the appropriate system of equations and solve it by using a computer algebra system, for example wxMaxima (andrejv.github.com/wxmaxima). The result will be in the form of the following fourth degree algebraic equation (2), the polynomial of which can always be factored into the product of two polynomials of the second degree

$$(ay^2 - axy - aby + bx^2)(ay^2 + axy - aby - bx^2) = 0. \qquad (2)$$

By analyzing these terms using the 'delta invariants' δ and Δ we find that, while the second factor of (2) always (for all values of a, b) represents a hyperbola (the solution identified by I. H.), the first factor defines an ellipse, hyperbola or parabola, depending on the relation between the values of parameters a, b (ellipse for $a < 4b$, hyperbola for $a > 4b$ and parabola for $a = 4b$).

4 Conclusion

Through particular examples from the 18th century book of geometry exercises we have learned how the methods of the analysis of conics and the search for loci have developed. We have also experienced that the return to historical tasks can be inspiring and beneficial if the solver is equipped with analytical methods and software such as GeoGebra and wxMaxima.

References

[1] I. Holfeld: *Exercitationes Geometricae.* Charactere Collegii Clementini Societas Jesu, Praha, 1773

[2] D. J. Struik: *A Concise History of Mathematics.* Courier Dover Publications, 1967.

[3] J. Zahradník: *Problémy z geometrie ve sbírce Ioannise Holfelda Exercitationes geometricae,* sborník 34. mezinárodní konference Historie matematiky, Poděbrady, 23. - 27. srpna 2013, Praha: Matfyzpress, 2013, str. 191

Study of historical geometric problems by means of CAS and DGS

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University of South Bohemia, Faculty of Education

July 4, 2014

OBJECTIVE OF THE LECTURE

To present several problems on loci and conics selected from a Latin book of geometrical problems that was printed in the 18th century in Prague.

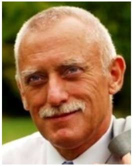
To show how the combination of DGS and CAS can help to find that published solutions are mostly incomplete and also to learn more about methods that are used in the book.

To present a computer as an effective tool to solve the problems with students in a both instructive and attractive way.

Outline

- Introduction of the book *Exercitationes geometricae*
- Introduction of the author *Ioannis Holfeld*
- Geometric properties used in the book
- Selected problems on loci - *Problema 24* & *Problema 25*
  - Original solution
  - Solution with the use of GeoGebra and wxMaxima
  - Utilization in the teaching of mathematics
- Conclusion

Collaborator

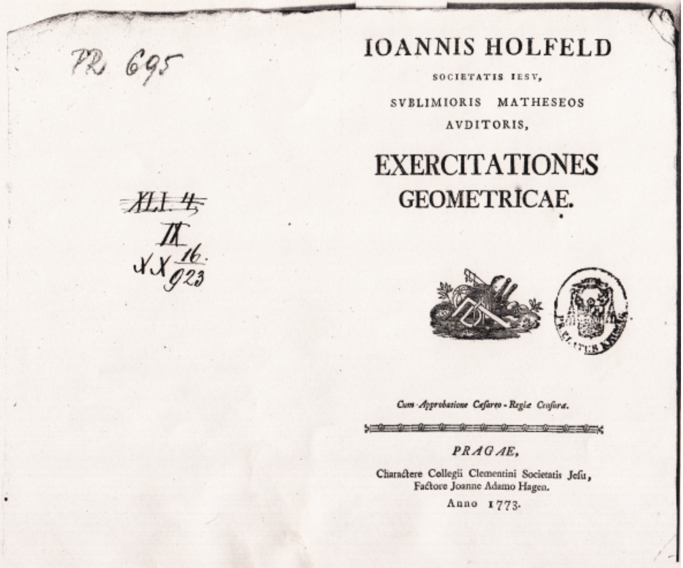


Jan Zahradník

University of South Bohemia, Czech Republic

Translation, interpretation of solutions, author's biography.

"Exercitationes geometricae"



IOANNIS HOLFELD
SOCIETATIS IESV,
SVBLIMIORIS MATHESEOS
AVDITORIS,

EXERCITATIONES
GEOMETRICAE.

Cum Approbatione Cæsareo - Regiæ Censuræ.

PRAGAE,
Charactere Collegii Clementini Societatis Jesu,
Factore Joanne Adamo Hagen.
Anno 1773.

PUBLISHER

Jesuit College of St. Clement in Prague, 1773

Deposition

Zlatá Koruna monastery
The Research Library of South Bohemia, Department of Historical Archives

CONTENT

65 pages, 47 solved exercises on geometry (illustrated by 36 figures):

- Problems on parabola (9).
- General problems on conics (13).
- Loci problems (19)
- Surfaces and volumes of solids of revolution (6).

CONTENT

26

Solutio: Sit BO axis Ellipſeos; AH tangat Ellipſim in E, & circulum in H; erit GH perpendicularis ad AH. A centro circuli dati duc rectam GD normalem axi BO. Semiordinata EC ſit $= y$; $CB = x$; $AC = z$; $AE = t$; $BD = c$; $GH = r$; $DG = d$; $AB = z - x$. Ob ſimilia triangula ACE, FHG, erit $z : t = r : FG$; $FG = \frac{tr}{z}$. Et quia ſimilia ſunt triangula ACE, ADF; erit: $z : y = z - x + c : FD$; & $FD = \frac{y(z - x + c)}{z}$. Eſt autem: $DF + FG = DG$; Proinde: $\frac{tr}{z} + \frac{y(z - x + c)}{z} = d$; Loco t, y, z, ponantur valores ex curvæ natura eruti, & functione abſciſſæ expreſſi. Constructa æquatione, ſeu: inventa abſciſſa BC ducatur ſemiordinata CE; & recta AE, tangens Ellipſim in E, tanget circulum.

Patet, Problema hac ratione ſolvi, ſi loco Ellipſis alia detur curva Algebraica.

PARS III.

DATA CVRVARVM GENESI, DETERMINANTVR NATVRÆ.

PROBLEMA 23.

31. Si datis duabus rectis AM, BM, (Fig. 25.) angulum quemcunque comprehendentibus, ſumptaque a puncto fixo A quavis parte AN, capiatur a fixo puncto B huic æqualis BO, & a puncto N ducatur NQ parallela rectæ MB, & occurrens rectæ AO in Q, invenire locum punctorum Q.

Solutio:

27

Solutio: Sit $AN = x$; $NQ = y$; $AM = a$; $MB = b$; erit: $x : y = a : b + x$. Hinc $ay = bx + x^2$; quæ eſt æquatio ad Parabolam. Eſt enim $ay + \frac{1}{4}b^2 = x^2 + bx + \frac{1}{4}b^2 = (x + \frac{1}{2}b)^2$. Fiat $x + \frac{1}{2}b = z$; erit $ay + \frac{1}{4}b^2 = z^2$. Ad a, & $\frac{1}{2}b$, quære tertium proportionalem terminum; qui ſit m; eritque $am = \frac{1}{4}b^2$; &: $ay + am = z^2$. Sit $y + m = u$; erit $au = z^2$; quæ eſt æquatio ſimpliciſſima ad Parabolam.

PROBLEMA 24.

32. Datis duabus rectis AM, MB, productaque ad libitum MB in O, ſumatur a puncto A recta AR æqualis MO; quæritur locus interſectionum rectarum RB, AO.

Solutio: Sit $AN = x$; $NQ = y$; $AM = a$; $MB = b$; erit $x : y = a : b + BO$; igitur: $BO = \frac{ay - bx}{x}$. Hinc: $RA = b + \frac{ay - bx}{x} = \frac{ay}{x}$; & $RN = \frac{ay + x^2}{x}$; $RM = \frac{ay + ax}{x}$. Eſt autem: $\frac{ay + x^2}{x} : y = \frac{ay + ax}{x} : b$. Nam: $RN : NQ = RM : MB$. Igitur: $y^2 + y(x - b) = \frac{bx^2}{a}$. Quæ æquatio ad ſimpliciſſimam ita reducitur: $y^2 + y(x - b) + \frac{(x - b)^2}{4} = \frac{(x - b)^2}{4} + \frac{bx^2}{a}$; Fiat $y + \frac{x - b}{2} = z$; erit $z^2 = \frac{(x - b)^2}{4} + \frac{bx^2}{a}$; & $4az^2 = ax^2 + ab^2 - 2abx + 4bx^2$; $4az^2 - ab^2 = x^2(a + 4b) - 2abx$.

Sive:

Content

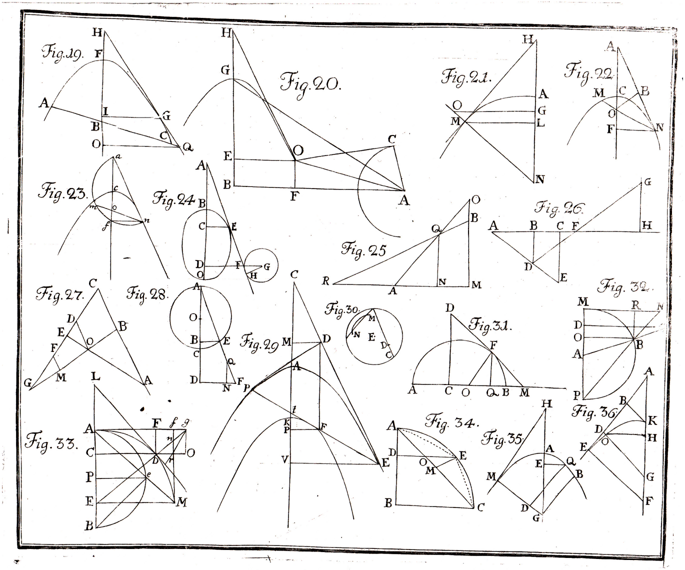


Fig. 19.
Fig. 20.
Fig. 21.
Fig. 22.
Fig. 23.
Fig. 24
Fig. 25
Fig. 26.
Fig. 27.
Fig. 28.
Fig. 29
Fig. 30.
Fig. 31.
Fig. 32.
Fig. 33.
Fig. 34.
Fig. 35.
Fig. 36.

Ioannis Holfeld

Historical sources: *Joannes Holfeld "Bohemus"*, *Johann Holfeld*

Possible biography:

- Born in 1750 in Poděbrady (Bohemia).
- In 1765 entered the Society of Jesus.
- In 1773 taught at the College of St. Clement in Prague.
- Taught mathematics at the University of Lviv (in Ukraine).
- Died in 1814 in Lviv.

Exercitationes Geometricae was probably his graduate thesis written under the supervision of Joannes Tessanek (1728 – 1788), Prague mathematician, physicist and astronomer.

GEOMETRIC PROPERTIES

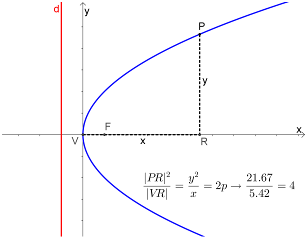
Methods of the book are based on the work of Apollonius.

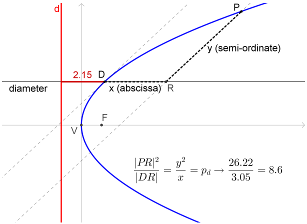
(*Introductio in analysis infinitorum* by Leonhard Euler that laid the foundations of analytical method was published in 1748)

GEOMETRIC PROPERTIES

- Coordinates

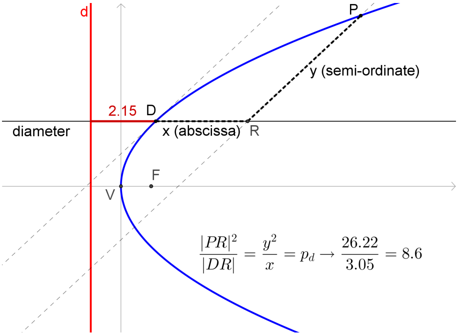
Instead of Cartesian coordinates are used conjugate diameters (*abscissa* and *ordinate*).





GEOMETRIC PROPERTIES

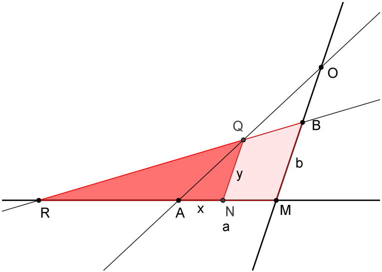
- Definition of a parabola



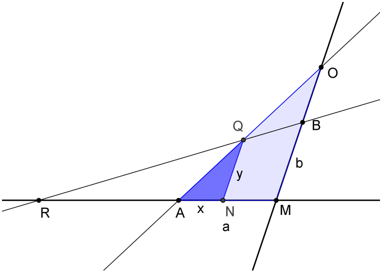
A parabola is defined as a curve for which the relation between the semi-ordinate PR and the corresponding abscissa DR of P is given by $PR^2 = p \cdot DR$.

GEOMETRIC PROPERTIES

- Similarity of triangles



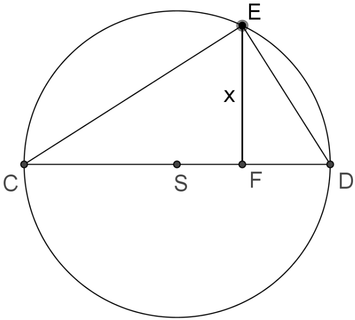
$\triangle RNQ \sim \triangle RMB$



$\triangle ANQ \sim \triangle AMO$

Geometric properties

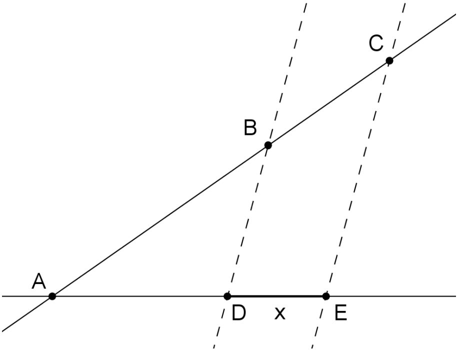
- Geometric mean theorem (Right triangle altitude theorem)



$$x = \sqrt{|CF| \cdot |FD|}$$

GEOMETRIC PROPERTIES

- Fourth proportional



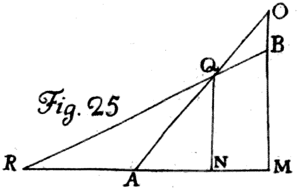
$$\frac{|AB|}{|AD|} = \frac{|BC|}{x}$$

$$x = \frac{|AD| \cdot |BC|}{|AB|}$$

PROBLEM 24

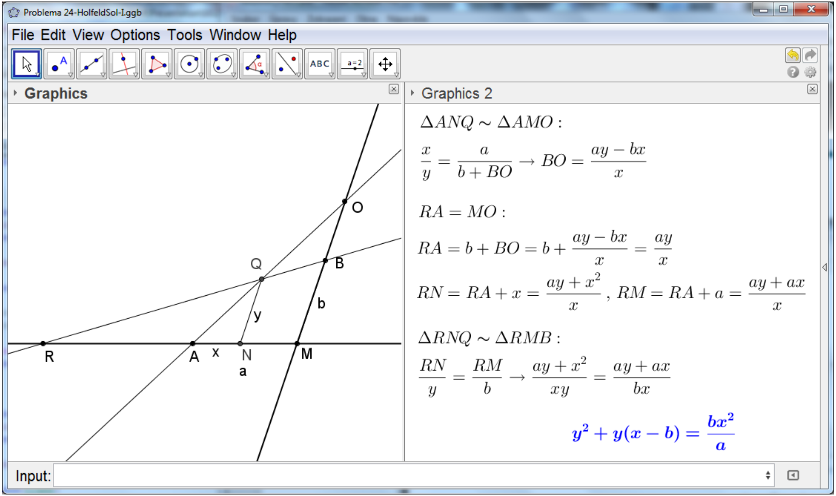
PROBLEMA 24.

32. Datis duabus rectis *AM*, *MB*, productaque ad libitum *MB* in *O*, ſumatur a puncto *A* recta *AR* æqualis *MO*; quæritur locus interſectionum rectarum *RB*, *AO*.



Problem: Given two lines *AM*, *MB* (see Fig. 25); select any point *O* on the line *MB*, construct the point *A* on the line *AR* so that *AR* is equal to *MO*; asking for the set of all intersections of two lines *RB*, *AO*.

Problem 24 - Holfeld's solution



Problem 24 - Holfeld's solution

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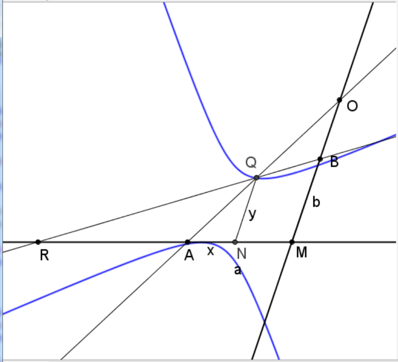
File Edit View Options Tools Window Help

Graphics



Graphics 2

$\Delta ANQ \sim \Delta AMO:$

$$\frac{x}{y} = \frac{a}{b+BO} \to BO = \frac{ay-bx}{x}$$

$RA = MO:$

$$RA = b + BO = b + \frac{ay-bx}{x} = \frac{ay}{x}$$

$$RN = RA + x = \frac{ay+x^2}{x},\ RM = RA + a = \frac{ay+ax}{x}$$

$\Delta RNQ \sim \Delta RMB:$

$$\frac{RN}{y} = \frac{RM}{b} \to \frac{ay+x^2}{xy} = \frac{ay+ax}{bx}$$

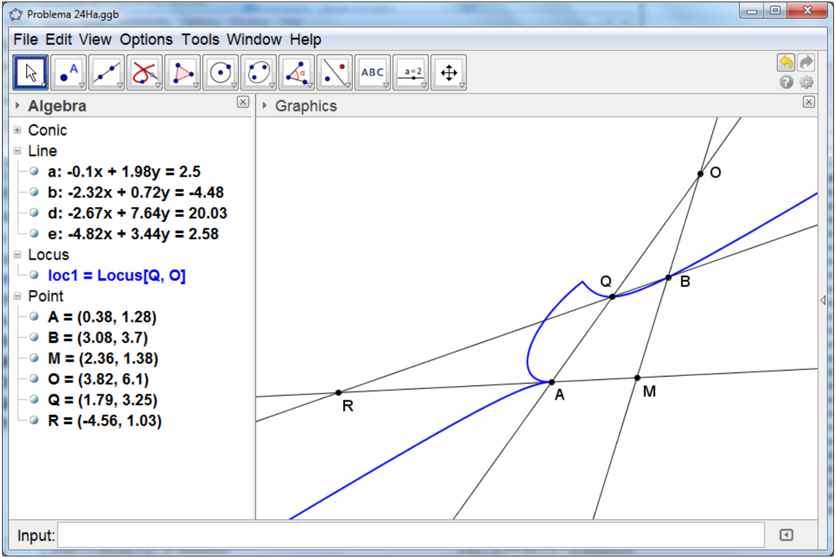
$$\mathbf{y^2 + y(x-b) = \frac{bx^2}{a}}$$

Input:

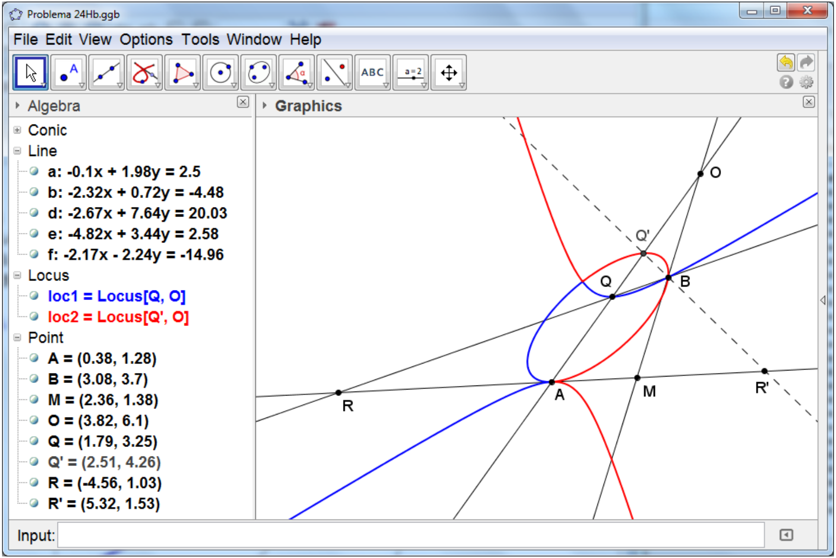
Problem 24

Is this a complete solution?

Problem 24 - The locus shape



Problem 24 - The locus shape

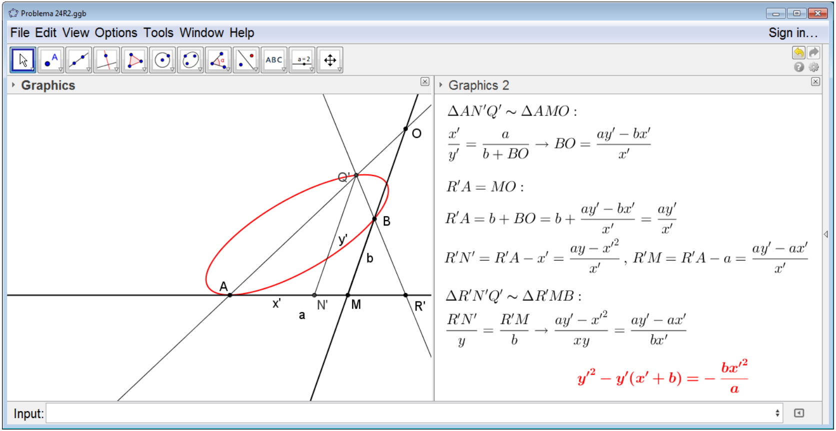


PROBLEM 24

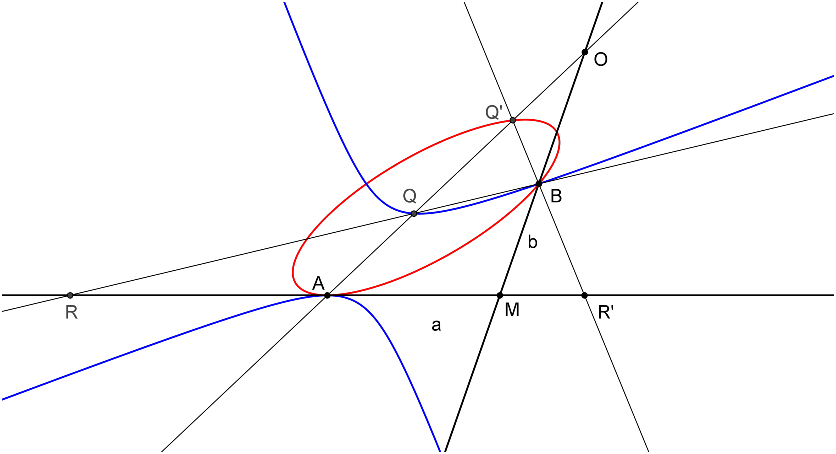
Why did I. H. not find the elliptical part of the locus curve?

Can this curve be found by means of the same procedure in which I. H. found the hyperbola?

PROBLEM 24 - MISSING SOLUTION



Problem 24 - Complete solution



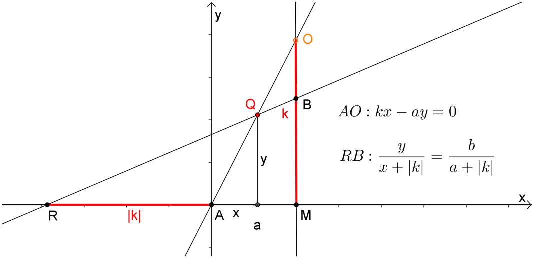
PROBLEM 24 - COMPLETE SOLUTION

Is the locus from the Problem 24 always consisting of a hyperbola and an ellipse?

Problem 24 - as a problem for undergraduates

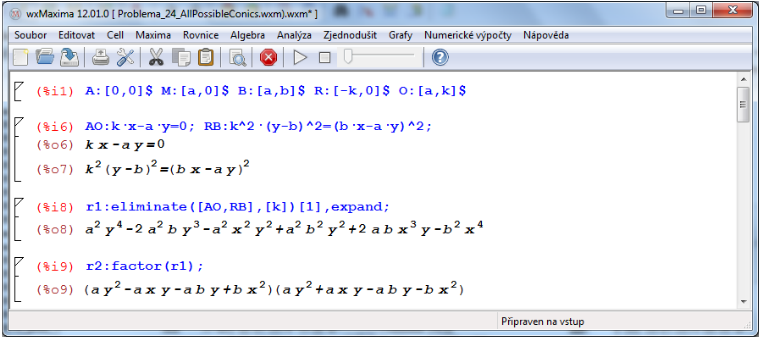
Problem: Find the locus equation to *Problem 24*. Discuss all the possible types of curves that form the locus.

Solution



$$AO: kx - ay = 0$$

$$RB: \frac{y}{x+|k|} = \frac{b}{a+|k|}$$



Solution

$$(ay^2 - axy - aby + bx^2)(ay^2 + axy - aby - bx^2)$$

The Delta invariants Δ, δ

$$ax^2 + 2hxy + by^2 + 2gx + fy + c$$

$$A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \delta = \begin{vmatrix} a & h \\ h & b \end{vmatrix}$$

$$\Delta \neq 0,$$

$$\delta > 0 : \text{ellipse}, \delta = 0 : \text{parabola}, \delta < 0 : \text{hyperbola}.$$

SOLUTION

(%i11) `k1:part(r2,1);`

(%o11) $a\,y^2-a\,x\,y-a\,b\,y+b\,x^2$

(%i12) `M1:M(k1); m1:submatrix(3,M1,3);`

(%o12) $\begin{bmatrix} b & -\frac{a}{2} & 0 \\ -\frac{a}{2} & a & -a\,b \\ 0 & -a\,b & 0 \end{bmatrix}$

(%o13) $\begin{bmatrix} b & -\frac{a}{2} \\ -\frac{a}{2} & a \end{bmatrix}$

(%i14) `D1:determinant(M1),factor;`
`d1:determinant(m1),factor;`

(%o14) $-a^2\,b^3$

(%o15) $\frac{a\,(4\,b-a)}{4}$

(%i16) `k2:part(r2,2);`

(%o16) $a\,y^2+a\,x\,y-a\,b\,y-b\,x^2$

(%i17) `M2:M(k2); m2:submatrix(3,M2,3);`

(%o17) $\begin{bmatrix} -b & \frac{a}{2} & 0 \\ \frac{a}{2} & a & -a\,b \\ 0 & -a\,b & 0 \end{bmatrix}$

(%o18) $\begin{bmatrix} -b & \frac{a}{2} \\ \frac{a}{2} & a \end{bmatrix}$

(%i19) `D2:determinant(M2),factor;`
`d2:determinant(m2),factor;`

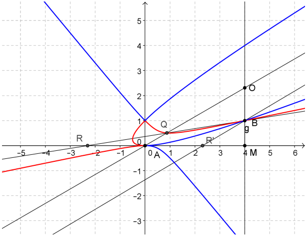
(%o19) $a^2\,b^3$

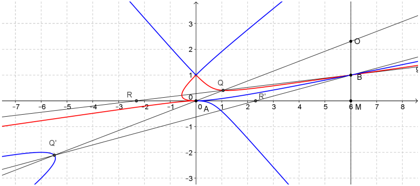
(%o20) $-\frac{a\,(4\,b+a)}{4}$

SOLUTION

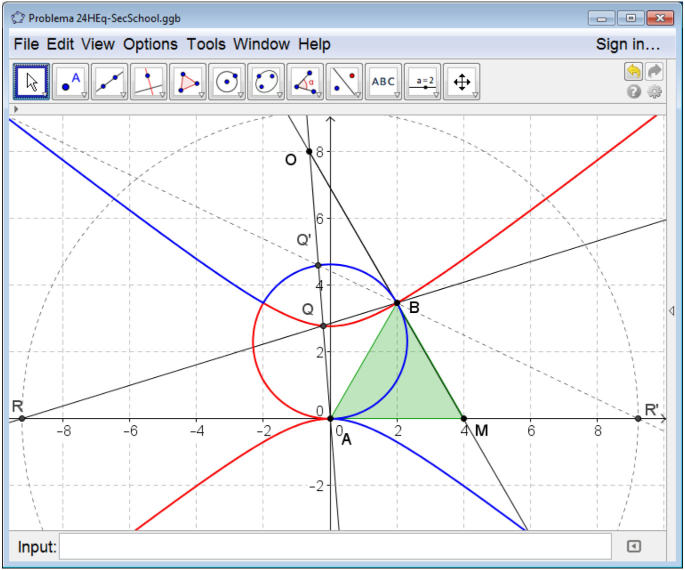
Components of the locus:

- Hyperbola,
- Ellipse ($a < 4b$) or Parabola ($a = 4b$) or Hyperbola ($a > 4b$).



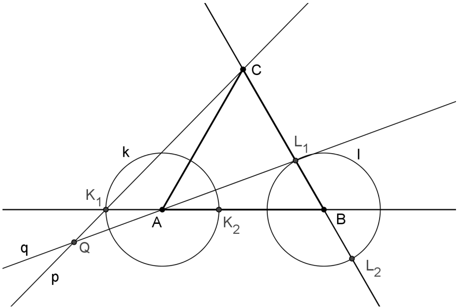


Problem 24 - as a secondary school problem



Problem 24 - as a secondary school problem

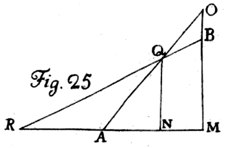
Problem: Given an equilateral triangle ABC and two circles k, l with centres in A and B, respectively, both with the radius r. Let K be the intersection of k and the line AB and L be the intersection of l and the line BC. Then p is a line given by points K and C and q is a line given by points L and A. Find the locus of point Q that is an intersection of lines p and q.



PROBLEM 25

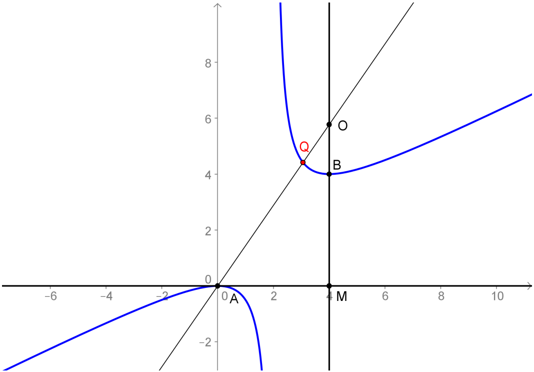
PROBLEMA 25.

35. Datis AM, MB, angulum rectum comprehendentibus, & ſumpta quavis BO, ductaque AO, quæ ſecetur in Q ea lege, ut ſit $OQ : QA = OB : OM$, invenire locum punctorum Q.

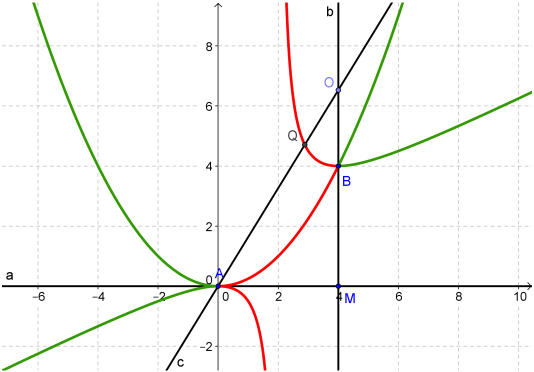


Problem: Given two perpendicular lines AM, MB (see Fig. 25); extend the line segment MB to a point O and construct the line AO, which contains the point Q so that $OQ : QA = OB : OM$. Find the locus of the point Q.

PROBLEM 25 - HOLFELD'S SOLUTION



PROBLEM 25 - COMPLETE SOLUTION



Conclusion

Return to the historical tasks, if the solver is equipped with analytical method and software like GeoGebra and wxMaxima can be inspiring and beneficial.

Utilization of the computer to solve these problems reveals the great benefits of todays mathematical software in the search for loci.

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