

## Inspirations from Outside

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**Abstract:** As a regular reader of journals, daily and weekly newspapers, digital newsletters, etc, I got many inspirations for treating the presented problems or ideas by means of a CAS. It is obvious that tables presented in scientific and economic journals inspire for graphic representation and statistic investigations. This will not be treated in this talk.

My examples shall demonstrate that one can find interesting and challenging problems for applying a CAS not only in textbooks. These problems can be for students of various mathematical levels and just for one's own entertainment. Among these examples are: modelling a baroque pillar with an interesting cross section, finding an analytic proof for a geometric problem, investigating the equation of a locus which was generated by a dynamic geometry program, modelling a special evolution strategy, and modelling the pricing of call & put options. Solving brain twisters discovered in journals or in respective collections of such problems with a CAS need other competencies than solving them by reasoning (see also my respective workshop).

A very special inspiration was given by working with LUA – a programming language which can be used with TI-NspireCAS. Clifford Pickover's "Mygalomorphs" get a new quality by applying sliders for changing shape and colours rapidly. This realisation of the "inspiration" was the result of a fruitful communication with a Swiss colleague.

In my demonstration I will use mainly DERIVE and TI-NspireCAS – because I am most familiar with these both systems.



Many teachers are fixed on the problems given in the textbooks or which can happen in the future on the kind of tasks given in the central exams.

It is my opinion that the math teacher should open the eyes and senses of the students for recognising mathematics in their environment and encouraging them feeling "inspired" by mathematical contents to treat them with their tools.

Over the years I found many inspirations for working with CAS – not only with my favourite means – the background pictures. I am a regular reader of several journals and weekly newspapers and receive a lot of newsletters from various software products.

You can expect a mathematical picture book with several chapters with not so detailed mathematical explications. If you would like to get more information then please feel free to contact me.



## Information Pages of Geometry

This problem was the **INSPIRATION** for my **INSPIRATIONS**-Lecture from today.

Find a parameter form for the surface of the depicted baroque pillar.

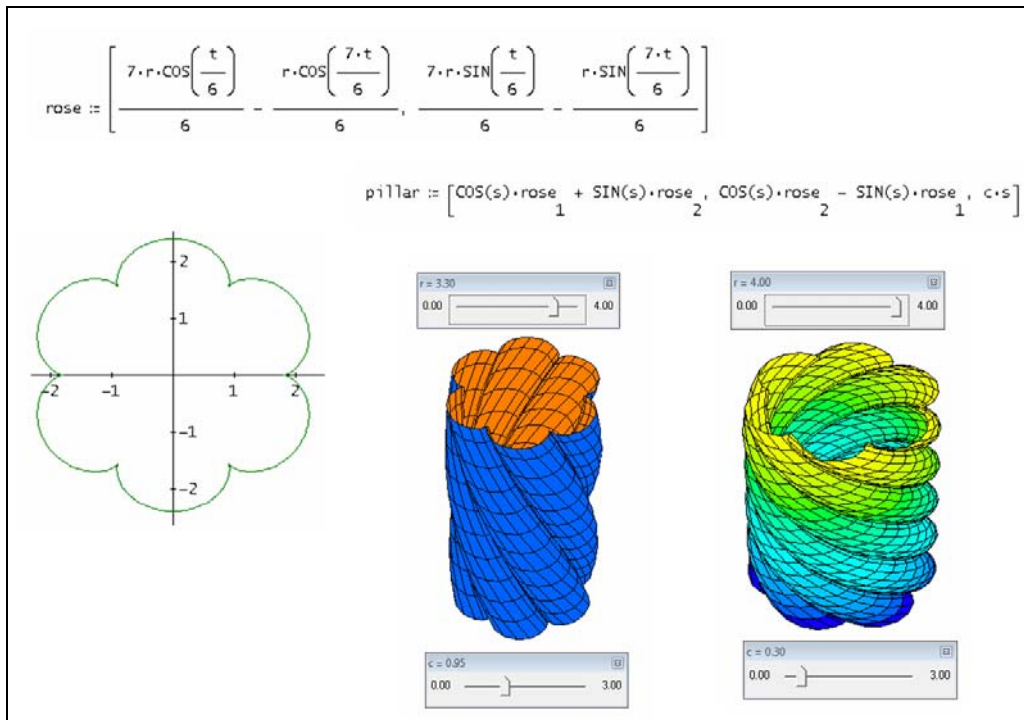
### Aufgabe 13.3 – schwierig



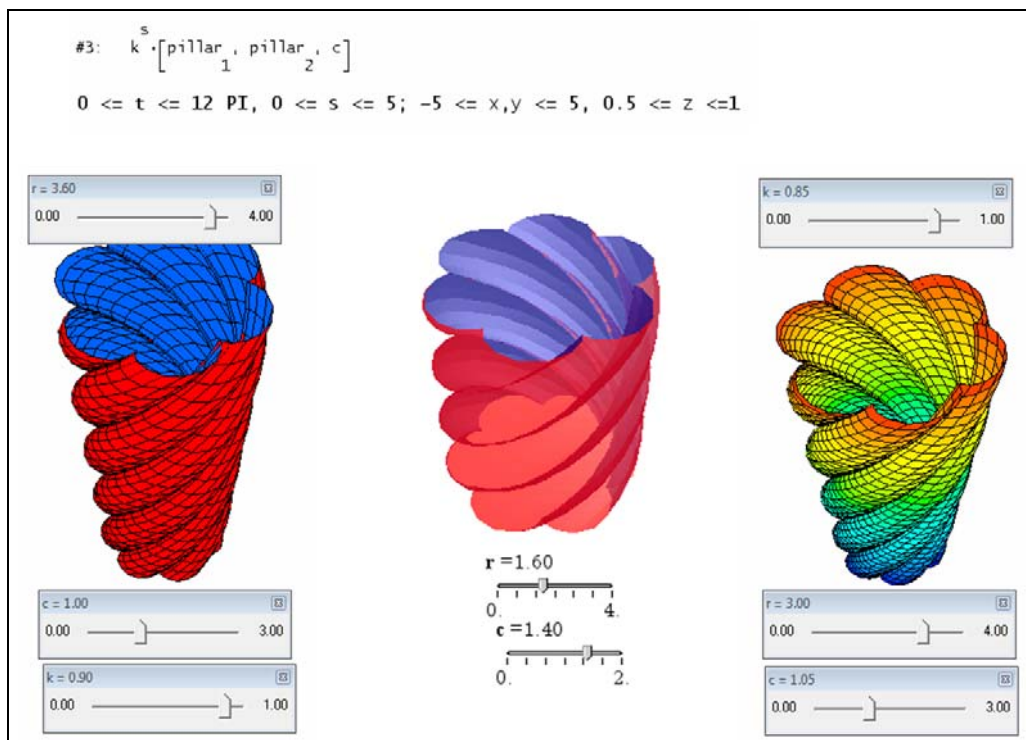
Man entwickle eine Parameterdarstellung für die Oberfläche der dargestellten Barocksäule.

The interesting part was modelling the cross section.

I took an epitrochoid and worked – not surprisingly – with sliders changing the parameters of the cross section curve and the screw surface.

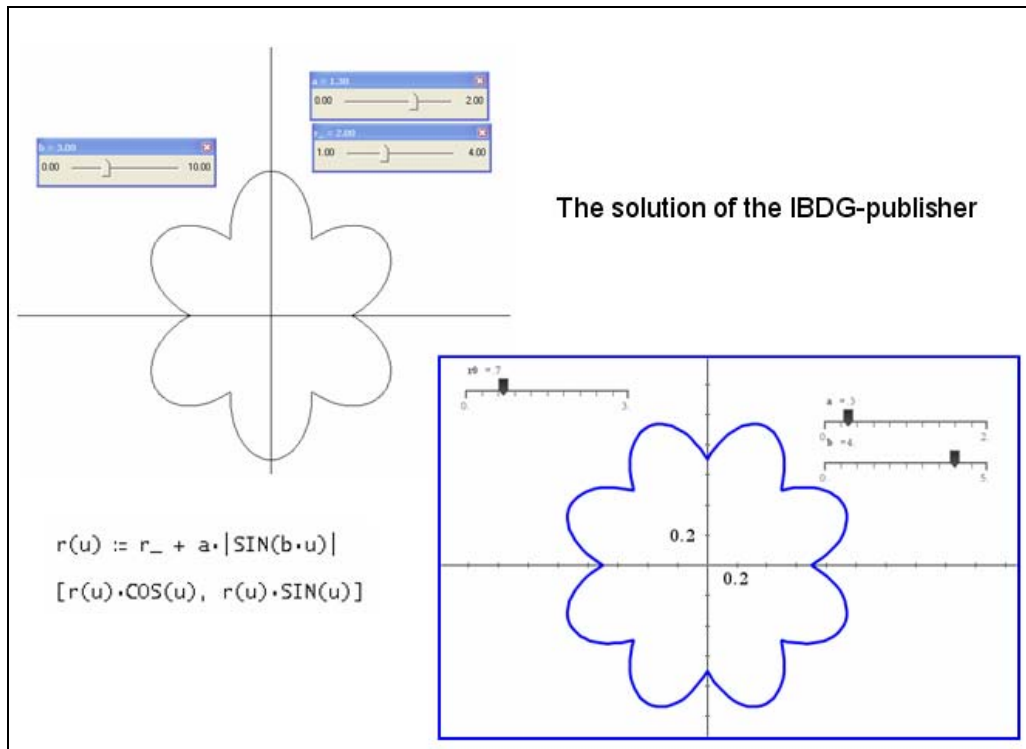


As the graph in the IBDG reminded me on a spiral surface I could not resist adding this solid, too. The centre picture is a screen shot from the Nspire (software) pillar.



I sent my solution to the editor of the IBDG – and it was published.

Gerhard Schröpfer, the editor, suggested another interesting way to generate the cross section. He strings a sine wave along a circle.



*Spektrum der Wissenschaft*, the German edition of *Scientific American*, is a rich resource for mathematical problems.

A special relation between triangles is investigated. A simple construction leads to a chaotic dynamic system.

GEOMETRIE UND CHAOSTHEORIE

### Dreiecksbeziehungen

Aus einer schlichten Konstruktionsvorschrift wird ein chaotisches dynamisches System mit den überraschendsten Eigenschaften.

VON GRÉGOIRE NICOLLIER

Es ist eigentlich ein Dreieck nichts weiter als eine Menge von drei Punkten in der Ebene – seinen Eckpunkten. Aber die klassische Geometrie kennt eine Fülle von Objekten, die man aus diesen drei Punkten konstruieren kann, sowie von Beziehungen dieser Objekte untereinander. Beispiele sind die speziellen Punkte aus der Schulgeometrie, in denen sich bemerkenswerterweise jeweils drei gleichartige Geraden schneiden: die Mittelsenkrechten, die Winkelhalbierenden, die Höhen und die Seitenhalbierenden.

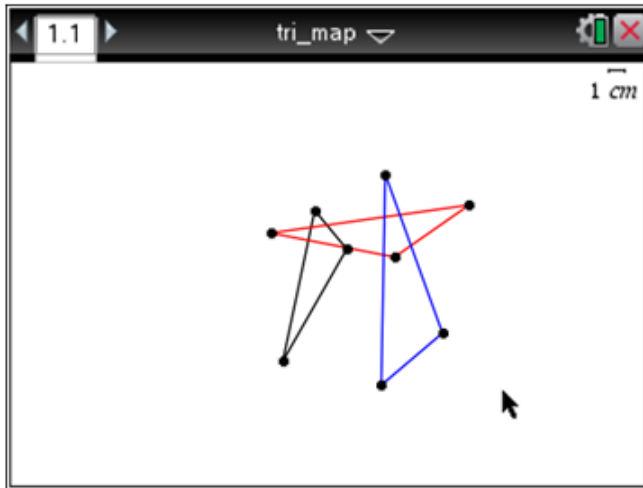
Entsprechend viele Möglichkeiten gibt es, aus einem Dreieck ein anderes zu machen. Man kann zum Beispiel als Eckpunkte des neuen Dreiecks die Mittelpunkte der Seiten des alten nehmen.

Nach 14 Schritten des Spiegelungsvorgangs ist das kleine rote Dreieck endlich spitzwinklig (gelbes Dreieck) und reist zu seiner gleichseitigen Grenzform weiter. Wie verläuft sein Weg?

Take any triangle in the plane and apply the following construction:

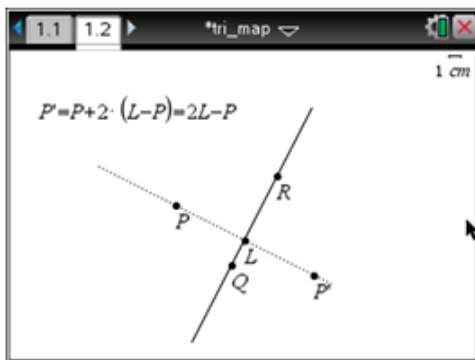
The vertices are reflected with respect to their opposite sides, giving the next triangle:

Black → Red → Blue



**3 steps on the Nspire Screen**

**Generalisation with DERIVE:**



$$g(Q, R) := y - R = \frac{R - Q}{2} \cdot \frac{x - R}{1}$$

$$n(Q, R, P) := y - P = - \frac{R - Q}{2} \cdot \frac{x - P}{1}$$

$$\text{SOLUTIONS}(g(Q, R) \wedge n(Q, R, P), [x, y])$$

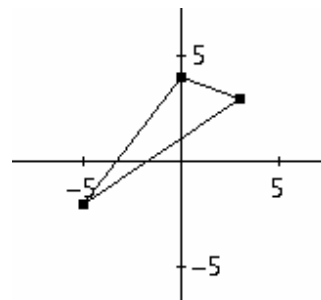
The position vector of P' (mapping of P) is given by a short vector calculation.

I generalise with DERIVE and we will inspect the DERIVE file and what will happen with the iterated triangles!

$$\text{trs}(A, B, C, n) := \text{ITERATES}\left[\begin{matrix} \text{PM}(v_1, v_2, v_3) & \text{PM}(v_2, v_3, v_1) & \text{PM}(v_3, v_1, v_2) & \text{PM}(v_1, v_2, v_3) \end{matrix}, v, [A, B, C, A], n\right)$$

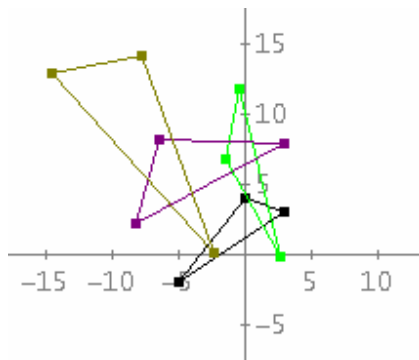
The initial triangle:

$$\begin{bmatrix} 3 & 3 \\ -5 & -2 \\ 0 & 4 \\ 3 & 3 \end{bmatrix}$$

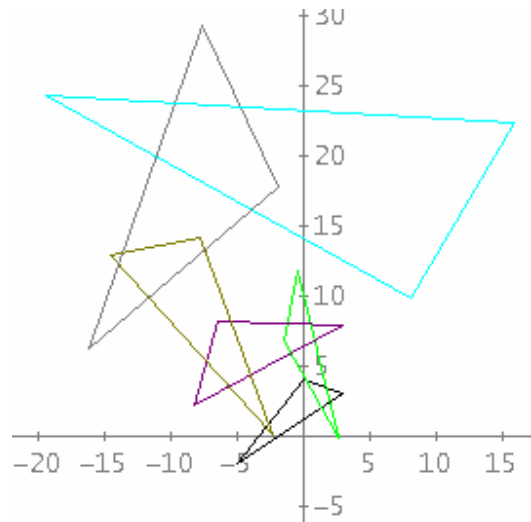


After three and after five iterations – the triangles become larger ...

`trs([3, 3], [-5, -2], [0, 4], 3)`

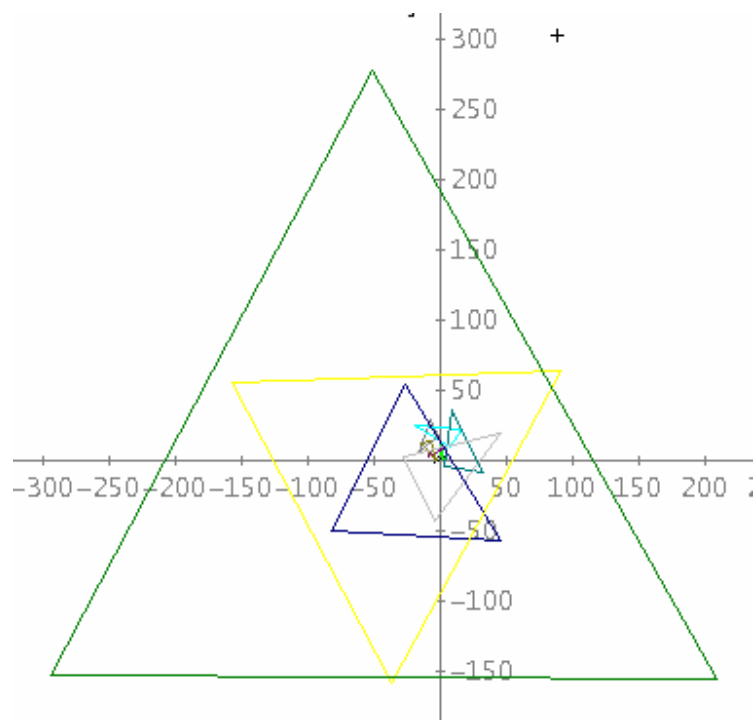


`trs([3, 3], [-5, -2], [0, 4], 5)`



.... and after 10 iterations:

`trs([3, 3], [-5, -2], [0, 4], 10)`

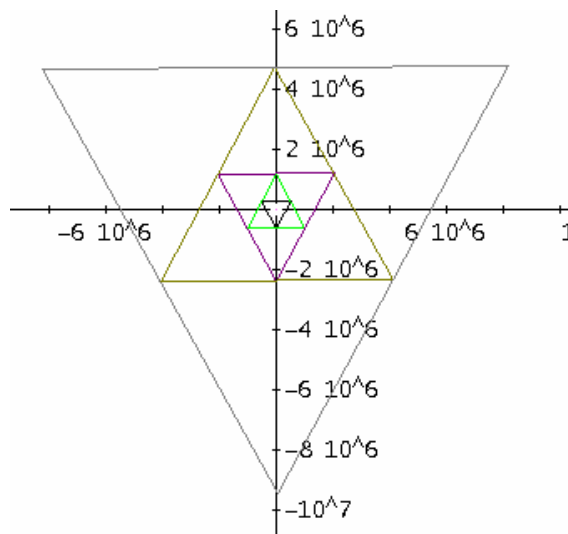


Now let me perform 25 iterations and inspect the last four triangles:

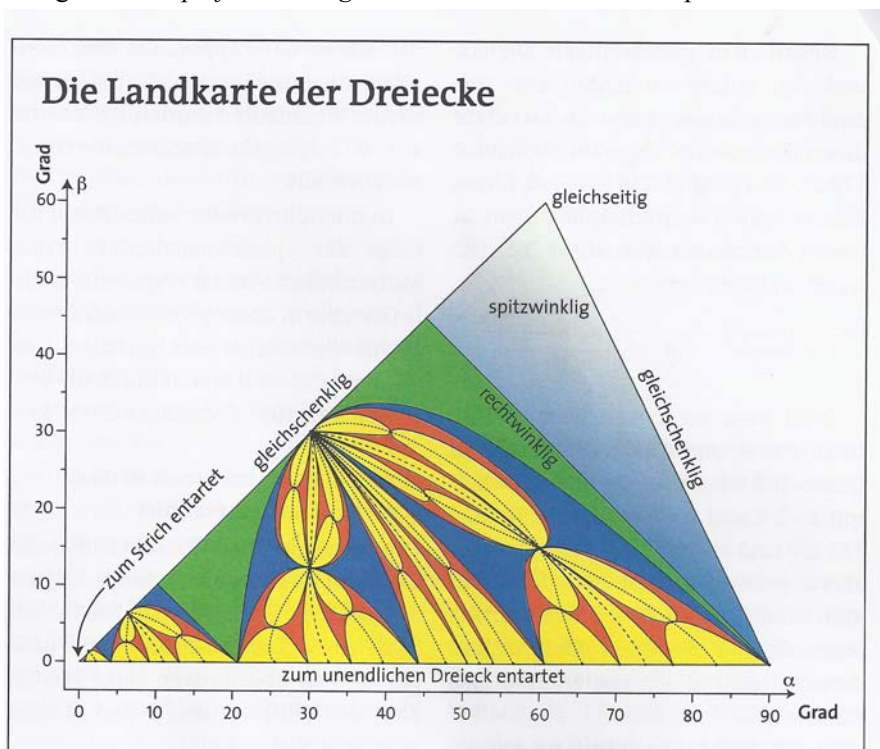
`(trs([3, 3], [-5, -2], [0, 4], 25))`

`[22, 23, 24, 25, 26]`

I have to zoom out very far to find them. The triangles tend to become equilateral!!!



In the *Spektrum* is given a *Map of the Triangles* which reminds on fractal patterns:



On the axes are two angles of the initial triangle (degree).

**green:** next step is an acute triangle – which tend to an equilateral triangle

**blue:** next step is a triangle in the green region

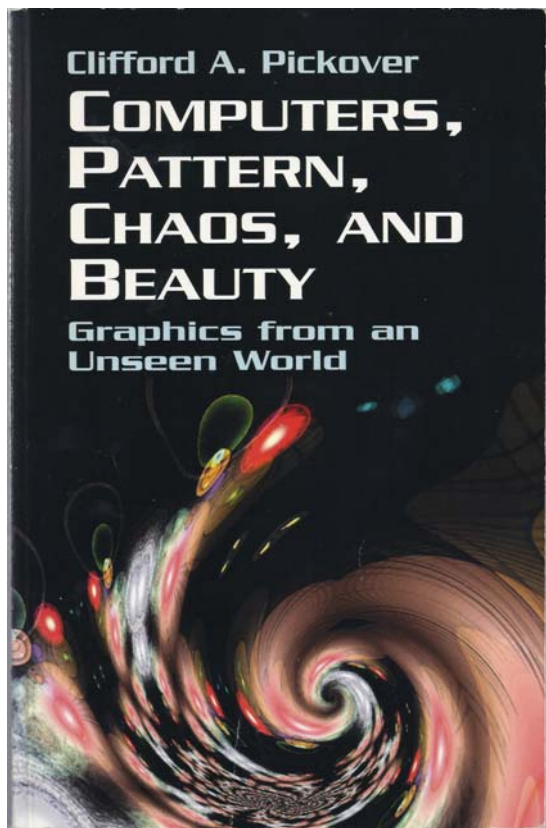
**red:** next step is a triangle in the blue region

**yellow:** next step is a triangle in the red region



From a German "Scientific American Special Issue" (1990).

I was – again – inspired by an article in an extra publication of the Scientific American where among many other inspirations A. K. Dewdney presented "Pickover's Delicacies".



I was always fascinated by his creatures – the BIOMORPHS but I never tried bringing them to life -

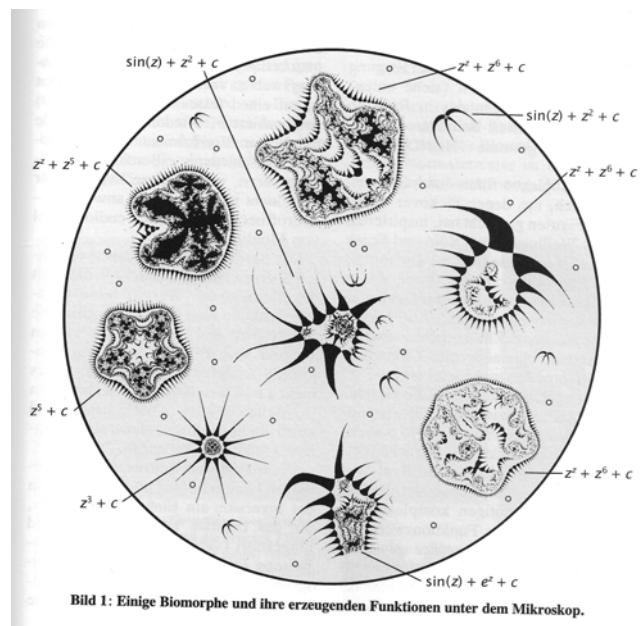


Bild 1: Einige Biomorphe und ihre erzeugenden Funktionen unter dem Mikroskop.

Pickover's Book is one of my – many – favourite books.

Small creatures - - a result of an programming error.

```

c ← .5 + .0i
FOR j ← 1 TO 100
  FOR k ← 1 TO 100
    berechne z0
    z ← z0
    FOR n ← 1 TO 10
      z ← z3 + c
      IF |real(z)| > 10 OR |imag(z)| > 10
        THEN verlasse die Schleife
      IF |real(z)| < 10 OR |imag(z)| < 10
        THEN zeichne(j,k) schwarz
        ELSE zeichne(j,k) weiß
  
```

It is an iteration of complex numbers e.g.  $z(n+1) = z(n)^3 + c$ .

We check if  $abs(real(z(n)))$  or  $abs(imag(z(n)))$  for  $n \leq 10$  exceeds the value of 10 then we will not plot the initial point, otherwise we will plot it. The initial complex numbers are within a given rectangle in the complex plane.

**Transfer from a Meta Code into a DERIVE program**



```

map(z) :=

biomorph(n, rmin, rmax, imin, imax, black, col, z0, j, k, dr, di, u) :=
  Prog
  black := []
  col := []
  dr := (rmax - rmin)/n
  di := (imax - imin)/n
  j := 0
  Loop
  If j > n exit
  k := 0
  Loop
  If k > n exit
  z0 := rmin + j.dr + i.(imin + k.di)
  z := z0
  "DISPLAY"
  u := 1
  Loop
  z := map(z, c)
  "DISPLAY(z)"
  If ABS(RE(z)) > 10 ∨ ABS(IM(z)) > 10 exit
  If u > 10 exit
  u := + 1
  If ABS(RE(z)) < 10 ∨ ABS(IM(z)) < 10
  Prog
  black := APPEND([[RE(z0), IM(z0)]], black)
  col := APPEND([u], col)
  k := + 1
  j := + 1
  black

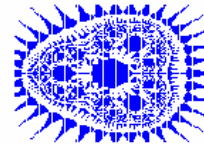
```

```


$$\text{map}(z) := z^3 + 0.5$$

biomorph(120, -1.5, 1.5, -1.5, 1.5)

```



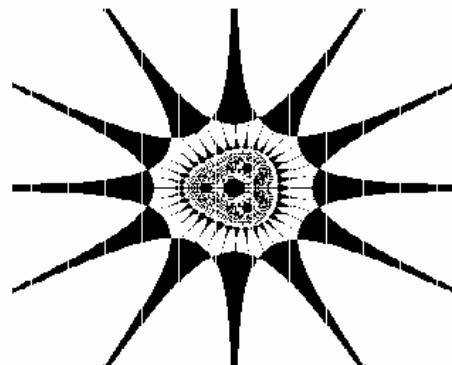
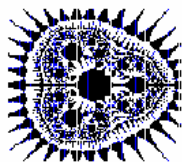
```
biomorph(300, -6, 6, -6, 6)
```

```

#6: 
$$\text{map}(z) := z^3 - 0.5$$

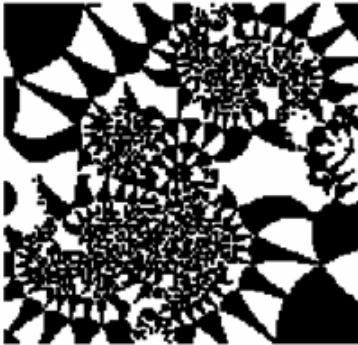
#7: biomorph(120, -1.5, 1.5, -1.5, 1.5)

```



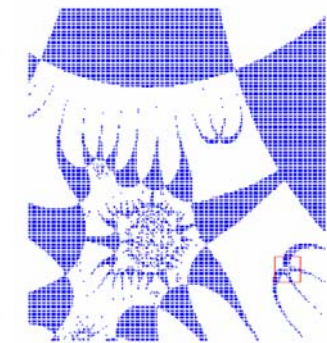
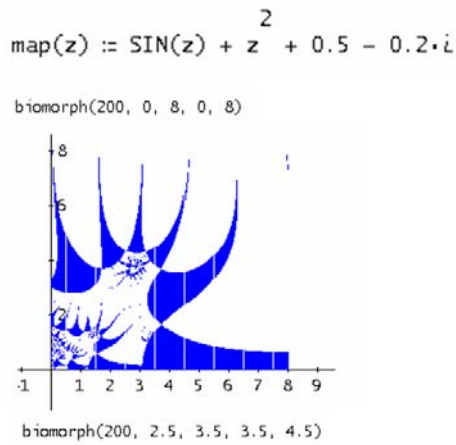
Some little beasts

```
#14: map(z) := SIN(z) + z2 + 0.5 - 0.2·i
#15: biomorph(200, -8, 0, 0, 8)
#16: biomorph(200, 0, 8, 0, 8)
#17: biomorph(200, -8, 0, -8, 0)
#18: biomorph(200, 0, 8, -8, 0)
```

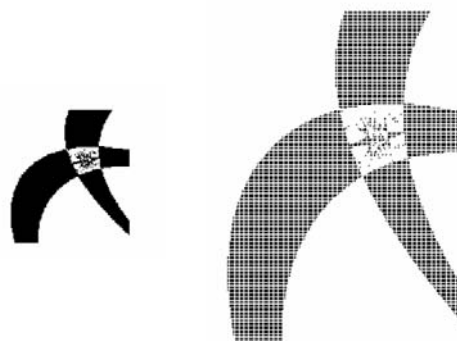
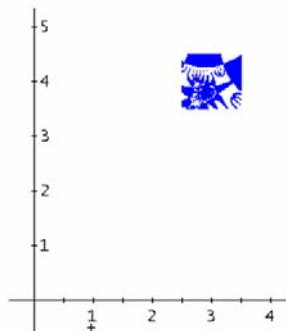


```
#19: map(z) := zz / SIN(z) + z2 - i
#20: biomorph(150, -1.5, 1.5, -1.5, 1.5)
```

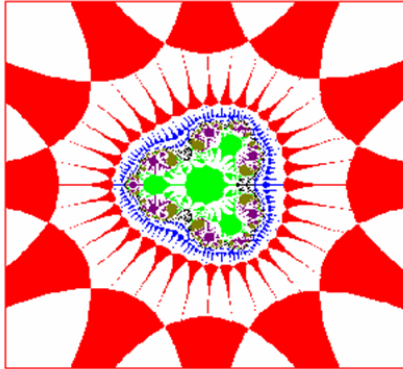
It is funny to extend the investigated part of the plane and to play around with combinations of functions. The fractal nature of the Biomorphs will become visible when we zoom in in special regions.



biomorph(200, 3.3, 3.4, 3.65, 3.75)



The interesting fact is that Pickover discovered the Biomorphs because he made a mistake in a program investigating several kinds of fractals.



$$\text{map}(z) := \frac{z^2}{\text{SIN}(z + i)} - \frac{1}{z} - i$$

biom(120, -3, 3, -3, 3)



## The Biomorph Zoo



My final "Inspiration" from the *Spektrum*.

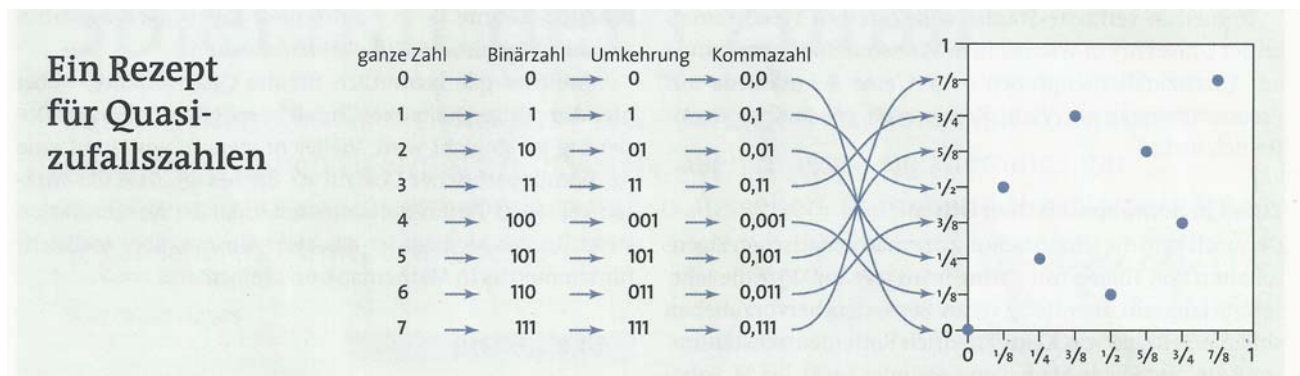
I will demonstrate Quasi Random Numbers.

They play among others an important role in financial Mathematics.



We will start with creating the van Corput Sequence – which is a nice application of number systems.

The next illustration describes the recipe for generating the first eight “quasi random numbers” which are fractions in [0,1).



The *DERIVE* realisation:

```

corput_(x, b := 2, p, o) :=
  Prog
  [o := 0, p := 1/b]
  Loop
  If x = 0
    RETURN o
  o := o + MOD(x, b)·p
  x := (x - MOD(x, b))/b
  p := p/b

```

```

corput_seq(n, b := 2) := VECTOR(corput_(k, b), k, n)

```

$$\text{corput\_seq}(10) = \left[ \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \frac{9}{16}, \frac{5}{16} \right]$$

Equidistant x-values and respective van Corput values on the y-axis give quasi random points within a given rectangle (from left bottom = lb to right top = rt)-

### From the Sequence to Quasi Random Points (Hammersley)

```

hammersley(n, lb := [0, 0], rt := [1, 1], b := 2, a_, b_, x_, y_) :=
  Prog
  [a_ := rt_1 - lb_1, b_ := rt_2 - lb_2]
  x_ := [VECTOR(lb_1 + i/n·a_, i, 0, n - 1)]
  y_ := [VECTOR(lb_2 + k·b_, k, corput_seq(n, b))]
  APPEND(x_, y_)

```

Taking another number base we get so called Halton Sequences and we can have two Halton Sequences on both axes resulting in a Halton set (= a set of quasi random points).

halton\_seq(n, b) := comput\_seq(n, b)

$$\text{halton\_seq}(10, 2) = \left[ \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \frac{9}{16}, \frac{5}{16} \right]$$

$$\text{halton\_seq}(10, 3) = \left[ \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \frac{1}{27}, \frac{10}{27} \right]$$

$$\text{halton\_seq}(10, 5) = \left[ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{25}, \frac{6}{25}, \frac{11}{25}, \frac{16}{25}, \frac{21}{25}, \frac{2}{25} \right]$$

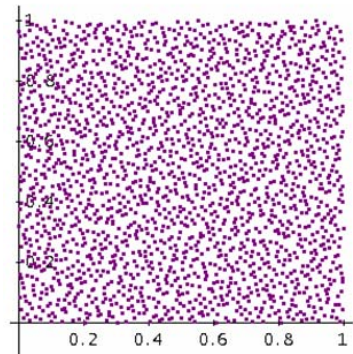
A Halton Set

halton\_seq(n, b) := comput\_seq(n, b)

$$\text{halton\_seq}(10, 2) = \left[ \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \frac{9}{16}, \frac{5}{16} \right]$$

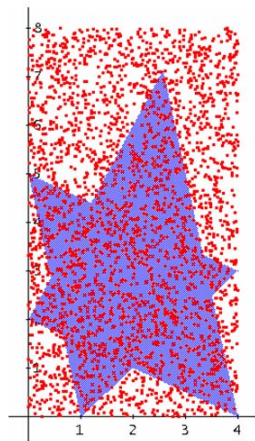
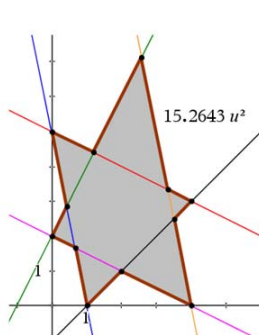
$$\text{halton\_seq}(10, 3) = \left[ \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \frac{1}{27}, \frac{10}{27} \right]$$

$$\text{halton\_seq}(10, 5) = \left[ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{25}, \frac{6}{25}, \frac{11}{25}, \frac{16}{25}, \frac{21}{25}, \frac{2}{25} \right]$$



Now let's apply these Quasi Random Points and compare with Random Points (Monte Carlo Method).

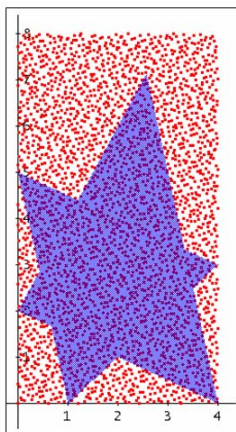
I don't take the example with the circle in order to find an approximation for PI.



VECTOR  $\left( \left| \frac{15.2643 - a}{15.2643} \right| \cdot 100, a, [15.0613, 15.36, 15.2213] \right)$   
 [1.329900486, 0.6269530866, 0.2817030587]

The maximum error is 1.33%.

Ordinary Monte Carlo Method



Errors are 0.2% and 0.1%. Convincing or not?

polpts := halton(3000, [0, 0], [4, 8], 2, 3)

DIM(SELECT(p(v<sub>1</sub>, v<sub>2</sub>), v, polpts)) = 1434

SOLVE  $\left( \frac{32}{a} = \frac{3000}{1434}, a \right) = (a = 15.296)$

$\left| \frac{15.2643 - 15.296}{15.2643} \right| = 0.002076741154$

polpts := halton(3000, [0, 0], [4, 8], 5, 11)

DIM(SELECT(p(v<sub>1</sub>, v<sub>2</sub>), v, polpts)) = 1433

SOLVE  $\left( \frac{32}{a} = \frac{3000}{1433}, a \right) = (a = 15.28533333)$

$\left| \frac{15.2643 - 15.2853}{15.2643} \right| = 0.001375759124$

I take a hexagon in the plane and would like to find its area (which is 15.2643 – calculated by Nspire in the Geometry-Application). The students could do it by applying analytic geometry or vector calculation.

I plot 3000 points into a circumscribed rectangle (three times applying MC) finding errors between 0.28% to 1.33%.

Two different Halton sets (again 3000 points) give the area with an error of 0.2% and 0.1%!!

This makes the difference!!

We can extend into space calculating the volume under a surface.

### What is the volume under the surface?

```

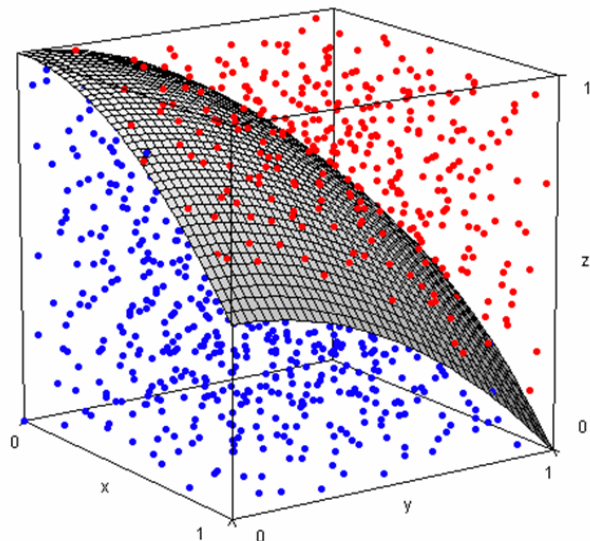
g(x, y) := 1 -  $\frac{x^2}{2}$  -  $\frac{y^2}{2}$ 

pts3d := halton3d(1000, [0, 0, 0], [1, 0, 0], [0, 1, 1], 2
DIM(SELECT(g(v1, v2) ≥ v3, v, pts3d)) = 667

 $\frac{667}{1000} = 0.667$ 

 $\int_0^1 \int_0^1 g(x, y) dx dy = \frac{2}{3}$ 

plot3dlist(SELECT(g(v1, v2) ≥ v3, v, pts3d))
plot3dlist(SELECT(g(v1, v2) < v3, v, pts3d))
    
```



The result is not so bad!!



I told about applications of quasi random numbers in financial mathematics. Doing some internet research I came across other algorithms for generating quasi random numbers as *Faure sequences* and *Sobol sequences*. The “nice” fact is that in most resources the same text is given. It seems to be that “Copy and Paste” is exercised in a very high degree.

The QRN given by a *Faure sequence* are product of an iteration process using an upper triangle matrix of binomial coefficients.

Among the many papers which I inspected on the web there was among others thesis containing an “interesting” upper triangular matrix of binomial coefficients. See the “thesis”-matrix and the correct one on the next page.

triangular matrix with elements where  $\binom{i}{j} = \frac{j!}{i!(j-i)!}$ :

$$\begin{bmatrix} \binom{0}{0} & \binom{0}{1} & \binom{0}{2} & \binom{0}{3} & \dots \\ 0 & \binom{1}{1} & \binom{1}{2} & \binom{1}{3} & \dots \\ 0 & 0 & \binom{2}{2} & \binom{2}{3} & \dots \\ 0 & 0 & 0 & \binom{3}{3} & \dots \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 2 & 3 & \dots \\ 0 & 0 & 1 & 3 & \dots \\ 0 & 0 & 0 & 1 & \dots \end{bmatrix}$$

$$\begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} \\ 0 & \binom{1}{1} & \binom{2}{1} \\ 0 & 0 & \binom{2}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

How to generate the 100<sup>th</sup> quasi random point in 4D-space using base 5:

Representation of 100 in base 5 is 400<sub>5</sub> which gives 0.004 = 4/125 = 0.032 (Halton sequence). This number is – represented as a vector [0,0,4] multiplied by this upper triangle matrix.

$$\text{ITERATES} \left( \text{MOD}(\text{trm}(3) \cdot v, 5), v, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, 3 \right) = \left[ \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} \right]$$

$$\left[ \frac{4}{125}, \frac{4}{5} + \frac{3}{25} + \frac{4}{125}, \frac{1}{5} + \frac{1}{25} + \frac{4}{125}, \frac{1}{5} + \frac{4}{25} + \frac{4}{125} \right]$$

$$\left[ \frac{4}{125}, \frac{119}{125}, \frac{34}{125}, \frac{49}{125} \right]$$

[0.032, 0.952, 0.272, 0.392]

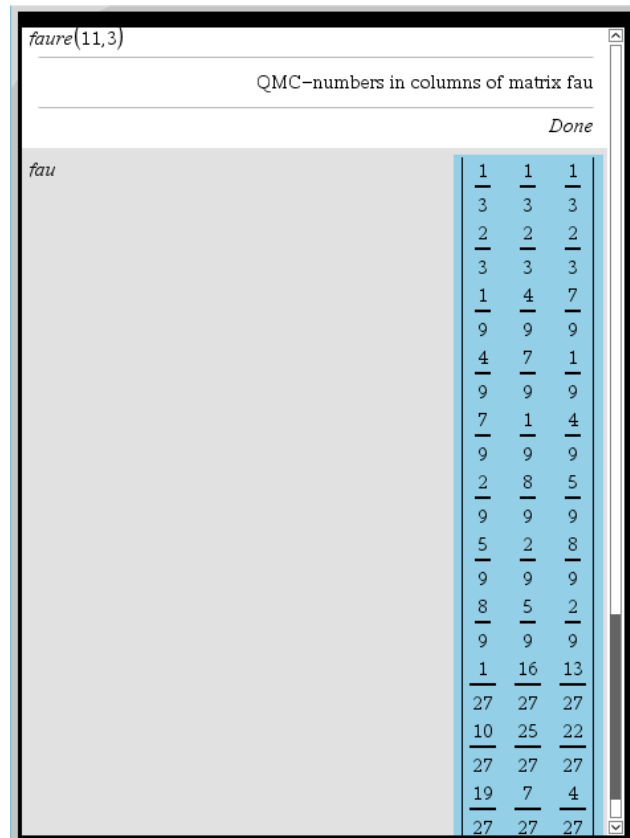
The screenshot shows a software interface with two panes. The left pane displays the execution of a Halton sequence generator. It shows the function call `halton(2000, [0 0], [pi/2 2], 3, 5)` and the resulting coordinates in lists `halx` and `haly`. Below this, it shows the count of iterations and the resulting matrix `fau` of QMC-numbers. The right pane shows the source code for the `faure` function, which generates the QMC-numbers by summing the digits of the input vector in base `p`.

**TABLE I**  
**THREE DIMENSIONAL FAURE SEQUENCES**

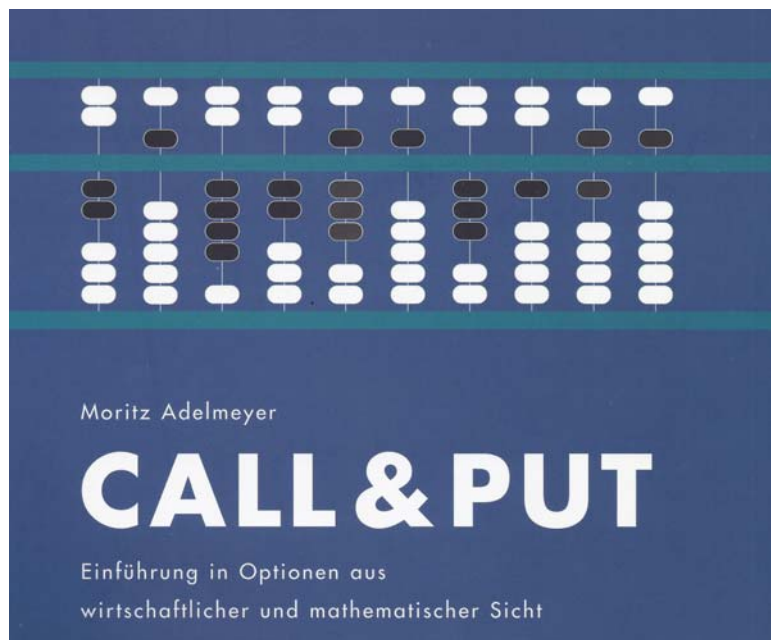
$n$	$a_0(n)$	$a_1(n)$	$a_2(n)$	$\phi_n^0$	$\phi_n^1$	$\phi_n^2$
1	1	0	0	1/3	1/3	1/3
2	2	0	0	2/3	2/3	2/3
3	0	1	0	1/9	4/9	7/9
4	1	1	0	4/9	7/9	1/9
5	2	1	0	7/9	1/9	4/9
6	0	2	0	2/9	8/9	5/9
7	1	2	0	5/9	2/9	8/9
8	2	2	0	8/9	5/9	2/9
9	0	0	1	1/27	16/27	13/27
10	1	0	1	10/27	25/27	22/27
11	2	0	1	19/27	7/27	4/27

We can compare the 3D *Faure* Sequences given in the Financial Mathematics paper and the TI-Nspire generated sequences.

There are also *Sobol*-sequences which I have not generated until now – another *Inspiration*?



Where to use these QRN will be shown in my next “Inspiration”.



**Introduction in Options from Economical and Mathematical Point of View**

A great introduction into Option Pricing (discrete and continuous model).



What are Options?

A call option gives the buyer of the option the right to buy the underlying asset at a fixed price (strike price or K) at any time prior to the expiration date of the option. The buyer pays a price for this right.

At expiration,

- If the value of the underlying asset (S) > Strike Price(K)
  - Buyer makes the difference: S - K
- If the value of the underlying asset (S) < Strike Price (K)
  - Buyer does not exercise

A put option gives the buyer of the option the right to sell the underlying asset at a fixed price at any time prior to the expiration date of the option. The buyer pays a price for this right.

At expiration,

- If the value of the underlying asset (S) < Strike Price(K)
  - Buyer makes the difference: K-S
- If the value of the underlying asset (S) > Strike Price (K)
  - Buyer does not exercise

A **STRANGLE** is purchase and selling of a Call- and Put option of the same stock with **different strike prices** but **same expiration dates**.

```
APPEND([[StockPrice, W/L Call, W/L Put, W/L Strangle]], TABLE([PayoffV_Call(21.5, 320, x),
PayoffV_Put(20, 280, x), PayoffV_Call(21.5, 320, x) + PayoffV_Put(20, 280, x)], x, 200, 440, 20))'
```

StockPrice	200	220	240	260	280	300	320	340	360	380	400	420	440
W/L Call	21.5	21.5	21.5	21.5	21.5	21.5	21.5	1.5	-18.5	-38.5	-58.5	-78.5	-98.5
W/L Put	-60	-40	-20	0	20	20	20	20	20	20	20	20	20
W/L Strangle	-38.5	-18.5	1.5	21.5	41.5	41.5	41.5	21.5	1.5	-18.5	-38.5	-58.5	-78.5

#28: PayoffV\_Call(21.5, 320, x) + PayoffV\_Put(20, 280, x)



The question is: How can we find the correct price of the options. (Options are bets on fall or rise of stocks and other goods like currencies, gold, and others.)

Take any stocks, say NoRiskComp stocks. What we need to know are the stock prices of the last periods (say 12 months). Then we can calculate the percentage change from one period to the other (rendite) and further the average profit and its volatility.

`kurse := [1665, 1820, 1676, 1730, 1785, 1942, 2210, 2149, 2193, 2142, 1825, 1897, 1835]`

`RENDITE(kurse) = [0.093093, -0.0791208, 0.0322195, 0.0317919, 0.0879551, 0.138002, -0.0276018, 0.0204746, -0.0232558, -0.147992, 0.039452, -0.0326831]`

`RENDITE_PROZ(kurse) = [9.3093, -7.91208, 3.22195, 3.17919, 8.79551, 13.8002, -2.76018, 2.04746, -2.32558, -14.7992, 3.9452, -3.26831]`

`AVERAGE(RENDITE_PROZ(kurse)) = 1.10278`

`VOLAT_PROZ(kurse) = 7.89387`

`rend := [9.3093, -7.91208, 3.22195, 3.17919, 8.79551, 13.8002, -2.76018, 2.04746, -2.32558, -14.7992, 3.9452, -3.26831]`

$$\sqrt{\frac{\sum_{i=1}^{\text{DIM}(\text{rend})} (1.10278 - \text{rend}_i)^2}{12}}$$

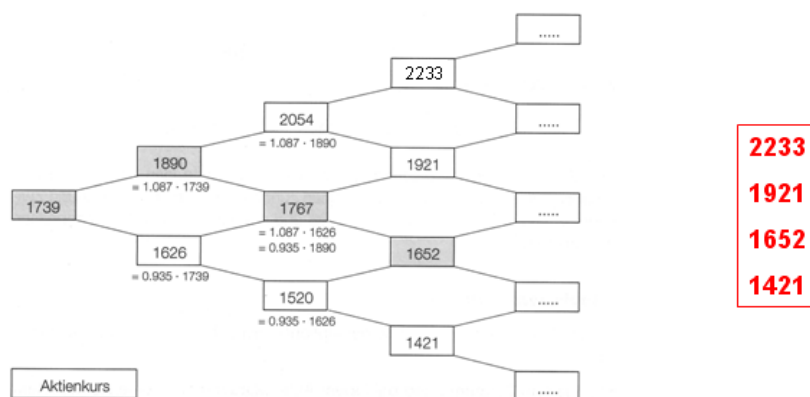
7.55781

We have mean  $1 \approx 1.1\%$  and volatility  $\approx 7.6\%$ .

We will generate the possible developments of the stock prices with *DERIVE* and we will inspect the possible prices at the end of the third period. Stocks can rise or fall from month to month:

### Possible stock prices after 1, 2, 3 or 4 periods

$\text{avg} = 1.1\%$ ,  $\text{volat} = 7.6\%$  →  $\text{avg} - \text{volat} \leq \text{rendite} \leq \text{avg} + \text{volat}$   
 $-6.5\% \leq \text{rendite} \leq 8.7\%$



Aktienkurs = stock price

**The CAS-Binomial Model**

Possible stock prices after 2, 3 or 4 periods based on estimations are:

$$\text{kurs\_binmod}(\text{kurs}, \text{avgrend}, \text{volat}, n) := \text{VECTOR}(\text{kurs} \cdot (1 + \text{avgrend} + \text{volat})^{n-k} \cdot (1 + \text{avgrend} - \text{volat})^k, k, 0, n)$$

$$\text{kurs\_binmod}(1739, 1.1\%, 7.56\%, 2) = [2053.236534, 1767.529407, 1521.578325]$$

$$\text{kurs\_binmod}(1739, 1.1\%, 7.56\%, 3) = [2231.046818, 1920.597454, 1653.347008, 1423.284365]$$

$$\text{kurs\_binmod}(1739, 1.1\%, 7.56\%, 4) = [2424.255473, 2086.921194, 1796.526859, 1546.540791, 1331.340195]$$

2233  
1921  
1652  
1421

What is the price of a call option on NoRiskComp stocks after 9 periods?

```
optpr(stpr, exerc, avgrend, volat, periods, r, v, klist, zlist, tabelle) :=
  Prog
  tabelle := []
  r := 1 + avgrend + volat
  v := 1 + avgrend - volat
  klist := VECTOR(rd2(stpr * r^k * v^(periods - k)), k, periods, 0, -1)
  zlist := VECTOR(IF(klist[k] > exerc, rd2(klist[k] - exerc), 0), k, periods + 1)
  tabelle := [klist, zlist]
```

```
optpr(1739, 1800, 0.003, 0.04, 9)
```

2540.14	740.14
2345.31	545.31
2165.42	365.42
1999.33	199.33
1845.98	45.98
1704.39	0
1573.66	0
1452.96	0
1341.51	0
1238.61	0

	06.03.	13.03.	20.03.
Stock price			2540
Option price			740
		2435	
		636	
	2335		2345
	536		545
		2249	
		449	
	2156		2165
	357		365
		2076	
		277	
	1991		1999
	192		199
		1917	
		117	
	1838		1846
	66.2		46
		1770	
		21.4	
	1697		1704
	10.0		0
		1634	
		0	
	1567		1574
	0		0
		1509	

2540 - 1800 = 740  
profit when exercised

The option price depends on the stock price and its parameters (avgrend, volat) and on the exercise (or strike) price.

The option price is calculated backwards by a simple linear system of two unknowns.

20. 02. XX:

We buy  $x_1$  stocks (stock price 1890) and invest  $y_1$  EURO at 0.125% per month.

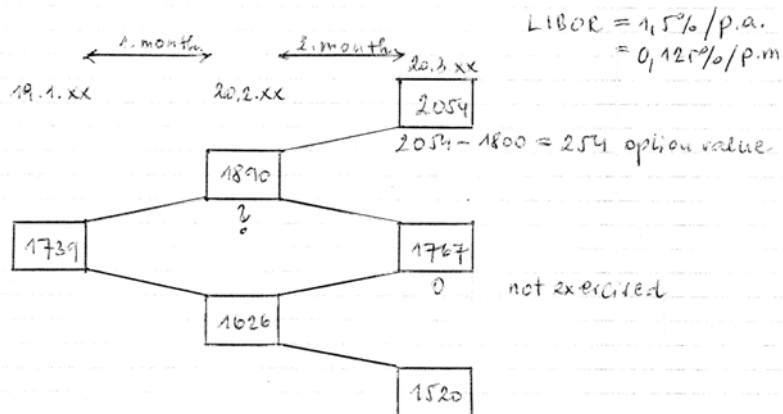
20. 03. XX: then we will have

either  $2054 x_1 + 1.00125 y_1 = 254$  (option value when exercised)

or  $1767 x_1 + 1.00125 y_1 = 0$  (option value – not exercised)

$x_1 = 0.8850, y_1 = -1562 \text{ €}$

Option value on 20. 03. XX:  $0.8850 \cdot 1890 - 1562 = 110.70$



One example from Call & Put

19.01.	23.01.	30.01.	06.02.	13.02.	20.02.	27.02.	06.03.	13.03.	20.03.
									2540
									0
								2435	
								0	2345
								2335	0
								0	2249
								2239	0
								0	2165
								2146	0
								0	2076
								2058	4.1
								0	1999
								1973	16.8
								0	1917
								1892	40.3
								1900	27.9
								1830	14.5
								1822	60.8
								1830	45.5
								1838	27.2
								1814	74.4
								1747	104
								1754	89.6
								1762	72.6
								1770	51
								1675	155
								1682	142
								1689	128
								1697	112
								1634	96
								1613	200
								1620	189
								1627	177
								1567	166*
								1553	250
								1560	241
								1567	233*
								1574	226
								1496	304
								1502	298*
								1509	291*
								1440	360*
								1447	353*
								1453	347
								1387	413*
								1393	407*
								1336	464*
								1342	458
								1298	514*
								1239	561

Put Option Price = 118

Stock Price = 1739

Exercise (Strike) Price = 1800

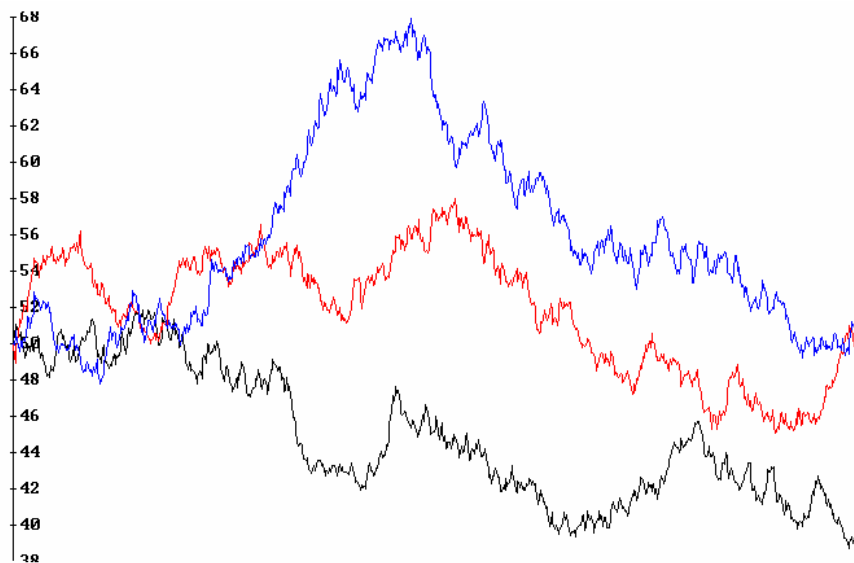
Can we do not only the calculation but also set up this option price tree wit a CAS? This is a challenge, indeed. But we can!!



Black & Scholes Formula is the continuous model – based on a more or less – random rise or fall of the stock price. The stock price follows a Brownian motion – see possible charts:

```
geombrown(y0, m, μ, σ, list, deltaT, i, x, y) :=
  Prog
    deltaT := 1/m
    list := [[0, y0]]
    x := 0
    y := y0
    i := 1
  Loop
    If i = m exit
    x := x + deltaT
    y := y*(1 + μ*deltaT + σ·RANDOM_NORMAL(1, 0)·√deltaT)
    list := APPEND(list, [[x, y]])
    i := i + 1
  RETURN list
```

```
geombrown(50, 800, 0.01, 0.4)
```



And this is the right moment to point to the quasi random numbers. The Monte Carlo Method using (pseudo) random numbers is substituted by applying QRNs.

```

BlackScholesC(kurs, ausueb, volat, zins, periods, d1, d2, r) :=
  Prog
    r := 1 + zins
    d1 := (LN(kurs*r^periods/ausueb) + volat^2*periods/2)/(volat*sqrt(periods))
    d2 := d1 - volat*sqrt(periods)
    RETURN NORMAL(d1)*kurs - NORMAL(d2)*ausueb/r^periods

```

```

BlackScholesP(kurs, ausueb, volat, zins, periods) := BlackScholesC(kurs,

```

$$\text{ausueb, volat, zins, periods) - kurs} + \frac{\text{ausueb}}{(1 + \text{zins})^{\text{periods}}}$$

$$\text{BlackScholesC}(1739, 1800, 4\%, 0.03\%, 9) = 59.414$$

$$\text{BlackScholesP}(1739, 1800, 4\%, 0.03\%, 9) = 115.561$$

**Put Option Price = 115.56**

**Stock Price = 1739**

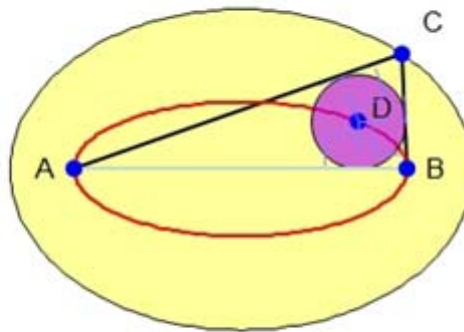
**Exercise (Strike) Price = 1800**

We have kept in mind 117.56 as result of the Binomial Model, looks quite good!!

Let me leave financial mathematics and turn to Geometry (Expressions)!

# Geometry Expressions Newsletter

## Problem of the Month



A string of length  $L$  is pinned at points  $A$  and  $B$ . Point  $c$  is the location of a pen, constrained by the string so that  $|AC| + |BC| = L$

The locus of  $C$  is, of course, an ellipse with foci  $A$  and  $B$  and major axis  $L$ .

What is the locus of point  $D$ , the incenter of the triangle  $ABC$ ?

[F := [-e, 0], G := [e, 0], P := [u, v]]

[equ1 := |P - F| = c, equ2 := |P - G| = 2·a - c]

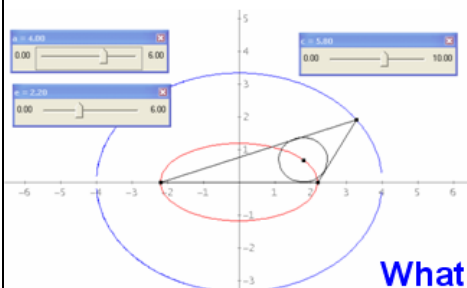
[equ1 :=  $\sqrt{(e^2 + 2·e·u + u^2 + v^2)} = c$ , equ2 :=  $\sqrt{(e^2 - 2·e·u + u^2 + v^2)} = 2·a - c$ ]

ang\_bis(a\_, b\_, c\_, t) :=  $b_ + \left( \frac{a_ - b_}{|a_ - b_|} + \frac{c_ - b_}{|c_ - b_|} \right) · t$

[f1 := ang\_bis(P, F, G), f2 := ang\_bis(F, G, P, s)]

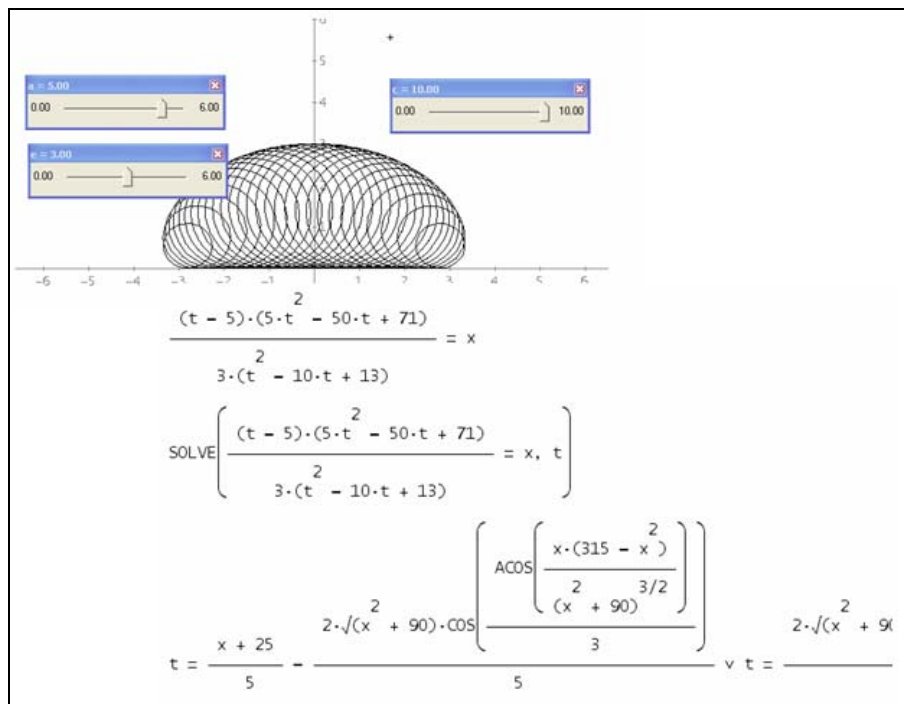
#32: The locus of the incircle centers is an ellipse.

$$\#33: \frac{x^2}{e^2} + \frac{y^2}{e \cdot (a - e)} = 1$$



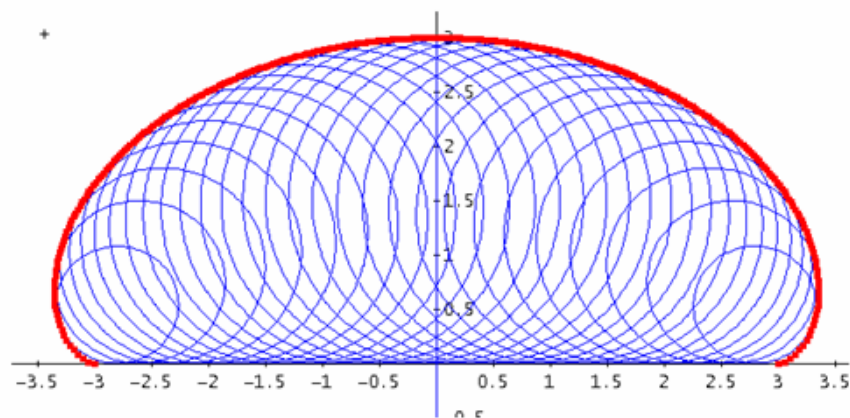
**Inspiration: I wanted more ...**

**What is the envelope of the circles?**



Again my beloved sliders. The envelope can be imagined but how does its equation look like?

$$\left[ \frac{(t-5) \cdot (5-t^2 - 50 \cdot t + 71)}{3 \cdot (t^2 - 10 \cdot t + 13)}, - \frac{4 \cdot (-t^2 + 10 \cdot t - 16)^{3/2}}{3 \cdot (t^2 - 10 \cdot t + 13)} \right]$$

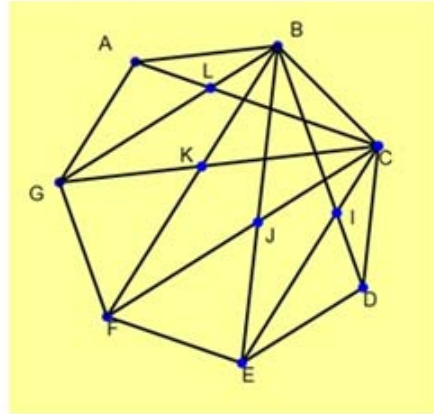


$$81 \cdot x^6 + x^4 \cdot \left( \frac{4617 \cdot y^2}{16} - 2187 \right) + x^2 \cdot \left( \frac{2673 \cdot y^4}{8} - \frac{10935 \cdot y^2}{4} + 19683 \right) + \frac{2025 \cdot y^6}{16} + \frac{10935 \cdot y^4}{16} - \frac{19683 \cdot y^2}{2} - 59049 = 0$$

It is a nice manipulation – using the power of a CAS – finding the implicit equation from the parameter representation. I would not like to do it by pencil & paper.



Another geometric challenge was this heptagon.



ABCDEFGH is a regular heptagon.

I is the intersection of BD and CE, J is the intersection of CF and BE, K is the intersection of CG and BF, L is the intersection of AC and BG.

Show that B,C,I,J,K,L are vertices of a regular heptagon.

An interactive illustration is [here](#).

Can you generalize this result?

Phil Todd: Yes, this is an example where the CAS may be counter-productive, as those general trigs are not at all easy for it to deal with.

My proof avoids lengths and stays in the angle domain...

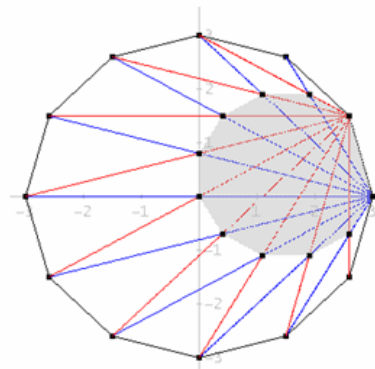
(Phil is the author of *Geometry Expressions*.)

```

po1(n) := VECTOR([a*cos(2*k*pi/n), a*sin(2*k*pi/n)], k, 0, n)
1f1(n) := VECTOR(1(n, 1, k, t), k, 3, n - 1)
1f2(n) := VECTOR(1(n, 2, k, s), k, 4, n)
po1new(n) := APPEND([ (po1(n))_2 ], VECTOR(SUBST(1f1(n), t, (SOLUTIONS(1f1(n))_k =
(1f2(n))_k, [s, t]))_1,2), k, 1, n - 3), [(po1(n))_1])
po1(12)
POLYGON_FILL(po1new(12))
    
```

Generalizing and plotting the HEXAGON.

But this is not a proof!!



Q1 is the intersection point of two corresponding diagonals in an n-gon with circumscribed circle = 1 and  $\text{diag}(n, i, j, p)$  is the segment connecting vertices  $\text{pt}(n, i)$  and  $\text{pt}(n, j)$  of the n-gon in parameter form (parameter p):

$$\text{pt}(n, k) := \left[ \cos\left(\frac{2 \cdot k \cdot \pi}{n}\right), \sin\left(\frac{2 \cdot k \cdot \pi}{n}\right) \right]$$

Finding the intersection point of any pair of corresponding diagonals

$$(\text{SOLUTIONS}(\text{diag}(n, 0, k, s) = \text{diag}(n, 1, k + 1, t), [s, t]))_1$$

$$\left[ \frac{\sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi \cdot k}{n}\right)}{2 \cdot \cos\left(\frac{\pi}{n}\right) \cdot \sin\left(\frac{\pi \cdot k}{n}\right)} + \frac{1}{2}, \frac{1}{2} - \frac{\sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi \cdot k}{n}\right)}{2 \cdot \cos\left(\frac{\pi}{n}\right) \cdot \sin\left(\frac{\pi \cdot k}{n}\right)} \right]$$

$$\text{Q1} := \text{SUBST}\left(\text{diag}(n, 0, k, s), s, \frac{\sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi \cdot k}{n}\right)}{2 \cdot \cos\left(\frac{\pi}{n}\right) \cdot \sin\left(\frac{\pi \cdot k}{n}\right)} + \frac{1}{2}\right)$$

$$\text{Q1} := \left[ \cos\left(\frac{\pi \cdot k}{n}\right)^2 - \frac{\sin\left(\frac{\pi}{n}\right) \cdot \sin\left(\frac{\pi \cdot k}{n}\right) \cdot \cos\left(\frac{\pi \cdot k}{n}\right)}{\cos\left(\frac{\pi}{n}\right)}, \frac{\sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi \cdot k}{n}\right)^2}{\cos\left(\frac{\pi}{n}\right)} + \sin\left(\frac{\pi \cdot k}{n}\right) \cdot \cos\left(\frac{\pi \cdot k}{n}\right) \right]$$

Q2 is the next intersection point:

$$\text{Q2} := \text{SUBST}(\text{Q1}, k, k + 1)$$

$$\text{Q2} := \left[ \frac{\cos\left(\frac{4 \cdot \pi \cdot k}{n} + \frac{5 \cdot \pi}{n}\right) - \cos\left(\frac{2 \cdot \pi \cdot k}{n} + \frac{3 \cdot \pi}{n}\right) + \cos\left(\frac{2 \cdot \pi \cdot k}{n} + \frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n}\right)}{2 \cdot \left(\cos\left(\frac{2 \cdot \pi \cdot k}{n} + \frac{3 \cdot \pi}{n}\right) + \cos\left(\frac{2 \cdot \pi \cdot k}{n} + \frac{\pi}{n}\right) - 2 \cdot \cos\left(\frac{\pi}{n}\right)\right)}, \right. \\ \left. \frac{\sin\left(\frac{4 \cdot \pi \cdot k}{n} + \frac{5 \cdot \pi}{n}\right) - \sin\left(\frac{2 \cdot \pi \cdot k}{n} + \frac{3 \cdot \pi}{n}\right) - \sin\left(\frac{2 \cdot \pi \cdot k}{n} + \frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)}{2 \cdot \left(\cos\left(\frac{2 \cdot \pi \cdot k}{n} + \frac{3 \cdot \pi}{n}\right) + \cos\left(\frac{2 \cdot \pi \cdot k}{n} + \frac{\pi}{n}\right) - 2 \cdot \cos\left(\frac{\pi}{n}\right)\right)} \right]$$

I (let) calculate the length of the segment Q1 Q2 and find that it does not depend on k:

$$|\text{Q1} - \text{Q2}| = \left| \tan\left(\frac{\pi}{n}\right) \right|$$

**This is a bundle of bulky trig expressions – but this is a nice result!!**

I didn't expect this pretty formula for the length of one side of the polygon!!

I can be sure that all edges of the polygon have equal lengths (nice result) but what about the angles? They should be equal, too.

Calculating the angle  $\angle Q1 Q2 Q3$ .

I need the next intersection point in the row, Q3:

$$Q3 := \text{SUBST}(Q2, k, k + 1)$$

In order to have more comfortable expressions I substitute  $\pi/n = \alpha \rightarrow n = \pi/\alpha$

$$Q1\_ := \text{SUBST}\left(Q1, n, \frac{\pi}{\alpha}\right)$$

$$Q1\_ := \left[ \cos(k \cdot \alpha)^2 - \frac{\sin(\alpha) \cdot \sin(k \cdot \alpha) \cdot \cos(k \cdot \alpha)}{\cos(\alpha)}, \frac{\sin(\alpha) \cdot \cos(k \cdot \alpha)^2}{\cos(\alpha)} + \sin(k \cdot \alpha) \cdot \cos(k \cdot \alpha) \right]$$

$$Q2\_ := \text{SUBST}\left(Q2, n, \frac{\pi}{\alpha}\right)$$

$$Q3\_ := \text{SUBST}\left(Q3, n, \frac{\pi}{\alpha}\right)$$

Finally DERIVE finds for me the angle formed by the two vectors  $[Q2, Q1]$  and  $[Q2, Q3]$  first with angle  $\alpha$ :

$$[r1 := Q2\_ - Q1\_ , r2 := Q2\_ - Q3\_]$$

$$\text{ACOS}\left(\frac{r1 \cdot r2}{|r1| \cdot |r2|}\right) = \text{SIGN}(\text{SIN}(2 \cdot \alpha)) \cdot \left(\frac{\pi}{2} - \text{MOD}(2 \cdot \alpha, \pi)\right) + \frac{\pi}{2}$$

$$2\alpha = 2\pi/n \text{ is definitely less } \pi \rightarrow \text{SIGN}(\text{SIN}(2\alpha)) = 1$$

$$\pi - \text{MOD}(2 \cdot \alpha, \pi)$$

Back substitution:  $\alpha = \pi/n$ , declaring  $n \geq 3$  and then simplifying:

$$n \in \text{Integer } [3, \infty)$$

$$\pi - \frac{2 \cdot \pi}{n}$$

which is exact the angle of the given n-gon.

All angles of the new polygon are equal.

**Our proof is complete and perfect!**

The MNU (=Mathematisch – Naturwissenschaftlicher Unterricht) is an organisation of German teachers for mathematics, information technology, science, chemistry, biology and geography.



**Die Evolutionsstabilität von Verhaltensstrategien**  
Dargestellt und verifiziert mit Hilfe einfacher Excel-Simulationen

ROLAND KÖPPL

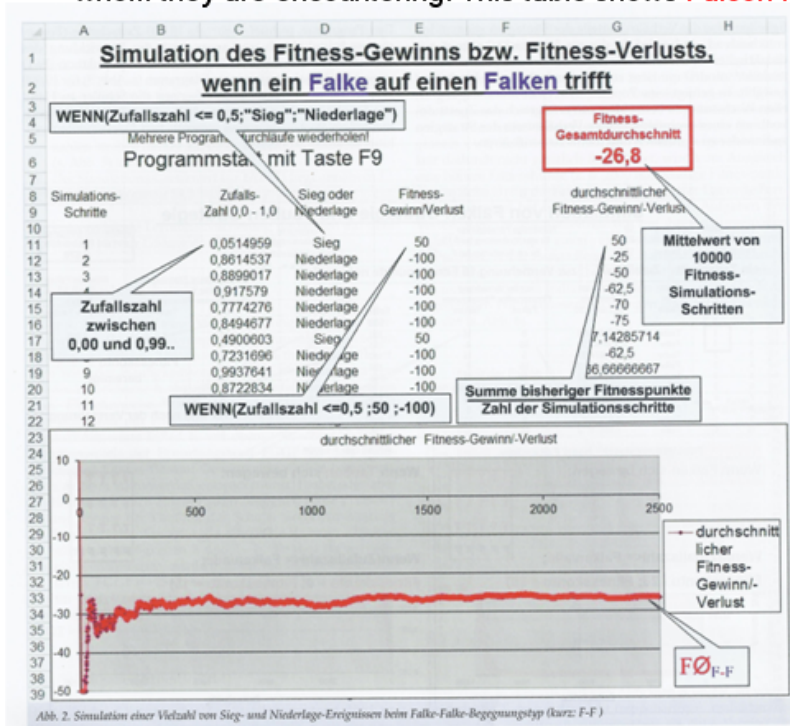
### Evolutionary Stable Strategies

A strategy which if adopted by all members of a population cannot be invaded by a mutant strategy through the operation of natural selection.  
(Note that a strategy is the behavioural phenotype of an individual, usually a composite of behaviours.)

<http://www.life.umd.edu/classroom/>



The birds can earn or lose **Fitness Points** depending on whom they are encountering. This table shows **Falcon meets Falcon**.



Expected Win/Loss of Fitness = -26.8

Roland Köppl, the author of the article sent an extended Excel-file for the simulations which inspired me to reproduce this simulation using CAS:

```

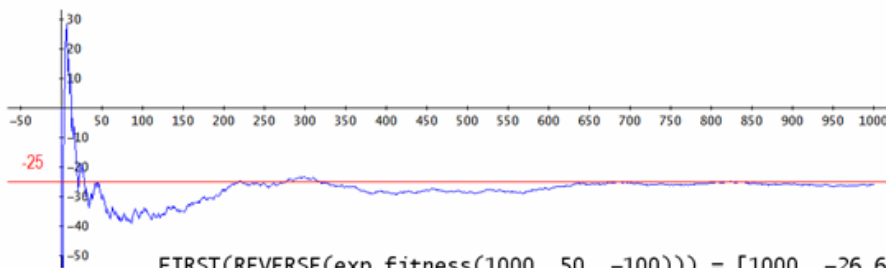
exp_fitness(n, wp, lp, i, fp, dummy, res, rd) :=
  Prog
  dummy := RANDOM(0)
  res := [[0, 0]]
  fp := 0
  i := 1
  Loop
  If i > n
  RETURN res
  rd := RANDOM(1)
  fp := fp + IF(rd ≤ 0.5, wp, lp)
  res := APPEND(res, [[i, fp/i]])
  i := i + 1
  
```

Falcon meets Falcon  
on the DERIVE screen.

Expected Fitness is -25

Expected Win/Loss = see below.

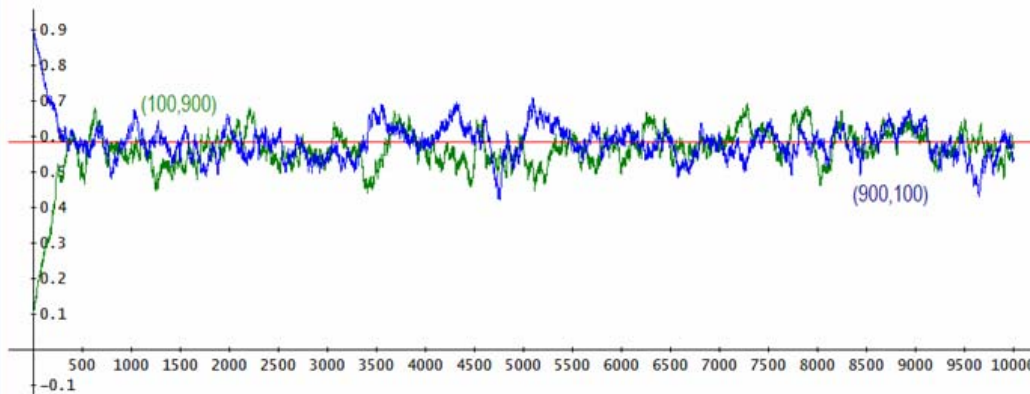
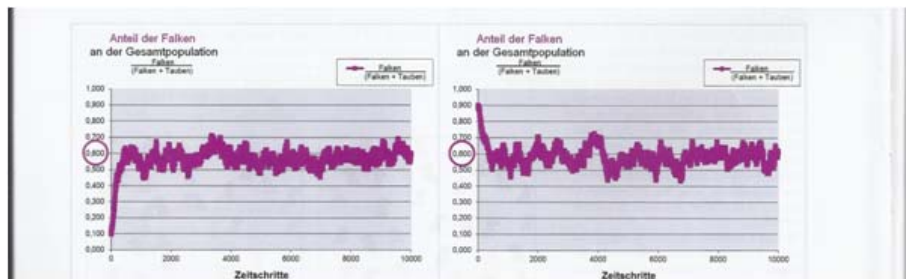
```
exp_fitness(1000, 50, -100)
```



FIRST(REVERSE(exp\_fitness(1000, 50, -100))) = [1000, -26.65]

FIRST(REVERSE(exp\_fitness(10000, 50, -100))) = [10000, -26.215]

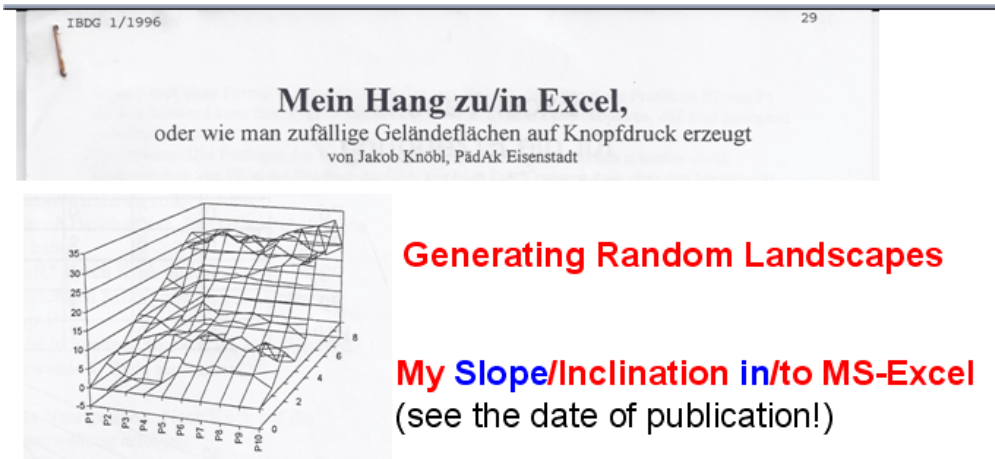
FIRST(REVERSE(exp\_fitness(10000, 50, -100))) = [10000, -25.375]



Percentage of Falcons in the whole population: Excel & DERIVE

<http://www.indiana.edu/~curtweb/S318/S318/lecturexi/lecturexi.html>

It was in 1996 when I attended a Excel-seminar given by a colleague from Burgenland. Together with numerous examples of Excel applications in financial mathematics and economics he demonstrated how to generate random landscapes.



Um eine Tabelle mit obigen Zufallszahlen zu erhalten, braucht man nur in zwei Zellen die richtigen Formeln schreiben und diese Formeln dann in die anderen Zellen kopieren.  
 Die Formel für Zelle C6 lautet:  $=B6+(AF-ZUFALLSZAHL())*MA$ . Damit wird zum Wert der links davon stehenden Zelle (hier B6) ein Vielfaches von MA addiert. Die Funktion Zufallszahl() liefert eine Zahl zwischen 0 und 1, sodaß der Faktor, mit dem MA multipliziert wird, für zB AF=75% zwischen -0,25 und 0,75 liegt.  
 Kopiert man die Formel aus C6 in die Zellen D6 bis J6, so erhält man eine geeignete Profilinie P1. Die Profilinie P2 kann nicht auf gleiche Weise erhalten werden, damit sie nicht unabhängig von P1 und damit die Geländefläche stark zerklüftet ist. Sie muß in engerem Zusammenhang zu P1 stehen, sodaß folgende Formel für die Zelle D6 vorgeschlagen wird:  $=C6+(0,5-ZUFALLSZAHL())*MA$ .

My *DERIVE* mountain range:

```
slope(ms, af, sf, n, h, hh, i, k, aux) :=
  Prog
    hh := []
    h := ITERATES(x + (af - RANDOM(1))*ms, x, 0, n)
    h := VECTOR([0, k - 1, h↓k], k, DIM(h))
    k := 1
  Loop
    If k > n + 1 exit
    aux := ITERATES([v↓1 + 1, v↓2, v↓3 + (0.5 - RANDOM(1))*ms*sf], v, [0, k - 1, h↓k↓3], n)
    hh := APPEND(hh, [aux])
    k := + 1
  COPROJECTION(hh)
```

Description of the program parameters:

$n$  is the number of grids ( $n \times n$ )

$ms$  is the maximal increase in the first profile (in the  $yz$ -plane)

$af$  is the "slope factor with  $0 \leq af \leq 1$  (1 = monotonous increasing, 0.5 = increasing or decreasing equally likely)

$sf$  describes the dependency of one profile line from the last one (from left to right).

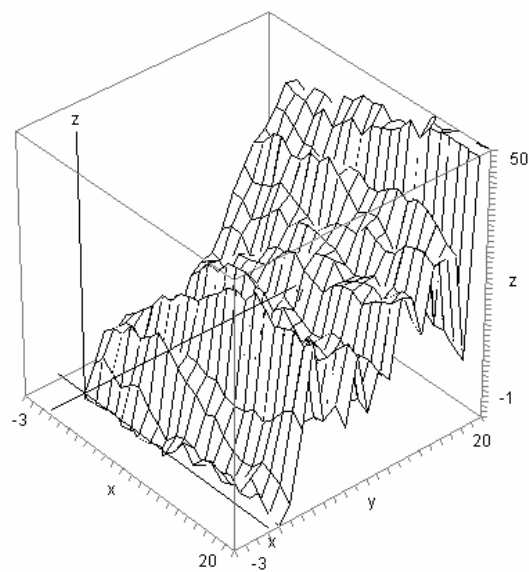
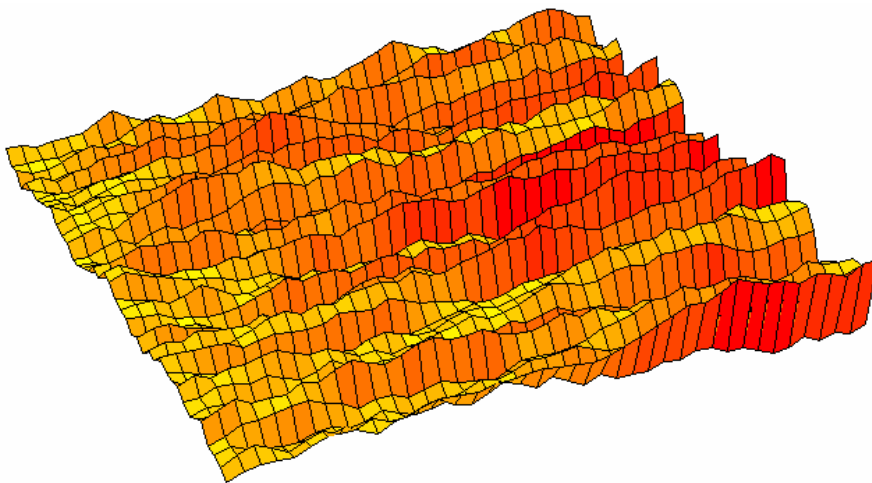
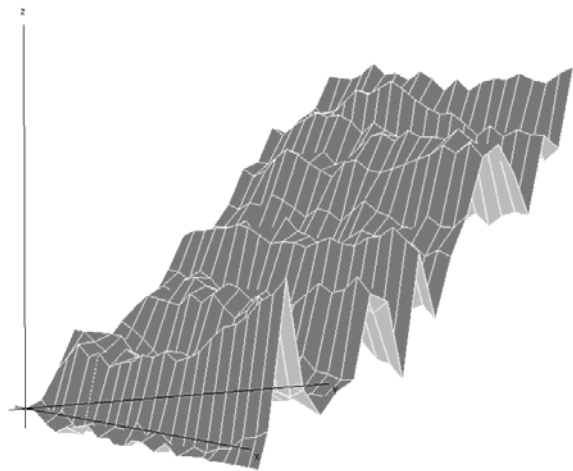
Good old COPROJECTION from *DERIVE* DOS times produces the complete grid of the landscape.

```
slope(3, 0.8, 0.5, 20)
```

```
slope(2, 0.5, 0.6, 40)
```

and another one which is steeper and wilder

```
slope(10, 0.8, 0.6, 20)
```



## ZEIT ONLINE LOGICALS

The weekly German journal *Die Zeit* presents in every issue a "Logical". These are brain twisters or brain teasers – puzzles which can be solved by logical reasoning. Then I found a lot of other resources offering problems which are nice to be tackled by means of CAS.

Family Wulffers spends the holidays on family Oxens farm. When Mrs. Oxens shows her farm to the four Wulffers children she asks "How old are you?".

Tanja laughs and answers: "Four years ago Hannah and I were of double age as Ralf and Paul were together."

Her twin sister Hannah proceeds: "And when Paul will be as old as Ralf is now then Ralf will be as old as I am now."

Mrs. Oxens looks despaired when Ralf says triumphantly: "When I will be one year older than Tanja is now then Paul will be three times as old than Hannah then will be."

Good Mrs. Oxens looks a little bit scared and Paul adds: "I have calculated that in twelve years our ages will sum up to exact 100 years."

Now Mrs. Oxens is completely confused and says: "Ok, one of you cannot calculate but if the statements of the other three are correct, then I do know your ages."

Who did not calculate correctly? How old are the four Wulffers children?

The solution will be given in my "Brain Twisters Workshop".

Another kind of problems are so called "Alphametiks":

Substitute the characters by integers such that the calculation becomes correct.

$$\begin{array}{r} \text{MAI} \\ + \text{JUNI} \\ + \text{JULI} \\ \hline \text{ALPIN} \end{array}$$

I worked with *DERIVE* because the Nspire's memory is "too small".

First of all I reanimated and adapted a program from an early *DERIVE* Newsletter to generate all  $k$ -order permutations (without repetition = variations) of  $n$  elements given in a list (vector).

See how this function works:

$$\text{perm}([A, E, X], 2) = \begin{bmatrix} E & X \\ X & E \\ A & X \\ X & A \\ A & E \\ E & A \end{bmatrix}$$



I define the four numbers according to the decimal places of the characters and the equation is the condition which must be fulfilled by the numbers.

```
[num1 := 100·m + 10·a + i, num2 := 1000·j + 100·u + 10·n + i]
[num3 := 1000·j + 100·u + 10·l + i, s := 10000·a + 1000·l + 100·p + 10·i + n]
cond := num1 + num2 + num3 - s = 0

[m := v1, a := v2, i := v3, j := v4, u := v5, n := v6, l := v7, p := v8]
all := perm([0, 1, 2, 3, 4, 5, 6, 7, 8, 9], 8)
```

As we have eight characters we need eight different numbers out of 0 through 9. all is the list of all possible permutations. How many?

The SELECT command finds out the right solution vector:

```
SELECT(cond, v, all) =
```

5	0	3	2	1	9	4	8
5	0	3	1	8	9	4	2
2	0	6	3	4	8	7	1
1	0	6	3	5	8	7	2
9	0	6	3	1	8	7	2
9	0	6	3	2	8	7	4
9	1	2	7	4	6	5	8

It seems to be that there are more solution. Inspecting all vectors we find out that the second element – which represents the character A – cannot be 0. So only the last combination is a possible solution:

```
SELECT(cond ^ v2 ≠ 0, v, all)
[[9, 1, 2, 7, 4, 6, 5, 8]]
needs 842 sec for how many checks?
COMB(10, 8)·8! = 1814400
```

**We see that A must be 1 or 2, let's try with A = 1:**

```
[m := v1, i := v2, j := v3, u := v4, n := v5, l := v6, p := v7, a := 1]
cond = (- 100·v7 - 990·v6 + 9·v5 + 200·v4 + 2000·v3 - 7·v2 + 100·v1 - 9990 = 0)
all := perm([0, 2, 3, 4, 5, 6, 7, 8, 9], 7)
(v := (SELECT(cond, v, all))) = v := [9, 2, 7, 4, 6, 5, 8]
```

**only 64 sec now!**

```
[num1, num2, num3, s] = [912, 7462, 7452, 15826]
COMB(9, 7)·7! = 181440
```

**912**  
**7462**  
**7452**  
**15826**

We can see that it can be helpful to mix two strategies: Reasoning and “CASing”.

I'd like to show you the puzzle which inspired me to look for more Teaser (in German: Knocheleien or Hirnverzwirner).

## 11 Da raucht der Kopf

Mathematik: -  
Schwierigkeitsgrad: 6

### Aufgabenstellung:

Schreiben Sie ein Programm, das folgendes Problem löst:

Abb. 11.1: "Da raucht der Kopf"

Jede Figur bedeutet eine Ziffer. Selbstverständlich sind gleiche Figuren auch gleiche Ziffern. Welche Ziffern sind an welche Stelle einzusetzen?


(Gerd Kebschull, Computer Knocheleien, Heise 1989)

(The Brain Twisters Workshop offers more examples and a collection of 50 problems in German and English as well).

Clifford Pickover appears a second time as "inspirator". Meet his *Mygalomorphs* which I met in MathPAD Online:

MathPAD Online

### Spider Nets - lines, procedures and mathematical art



## 2. The story of the Mygalomorph

It might be a bit difficult to trace back who and when invented the story of the Mygalomorph. I found it in a very interesting book "Keys to infinity" by Clifford A. Pickover (see [3]). However, he mentions other publications, which are based on some other publications, and so on. So, let us concentrate on the story of the Mygalomorph and draw from it what we need for this article - the concept of linear patterns, which in this article we will call spider nets.

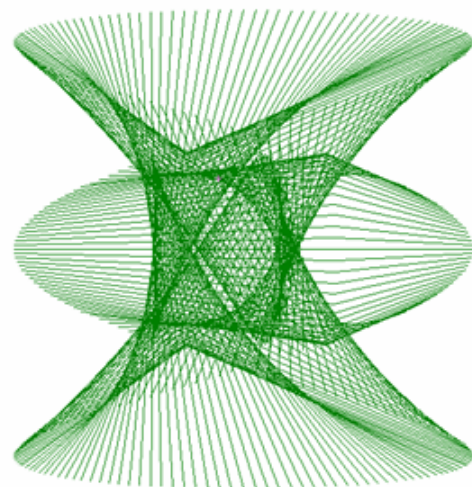
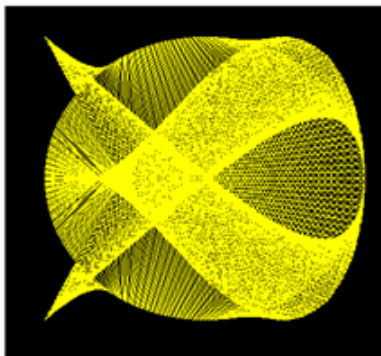
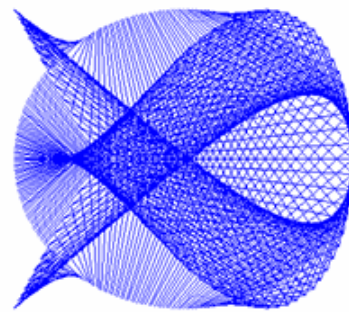
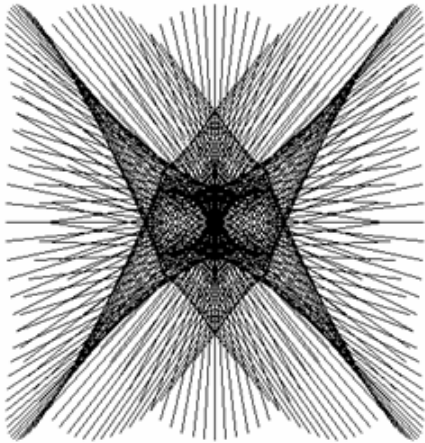
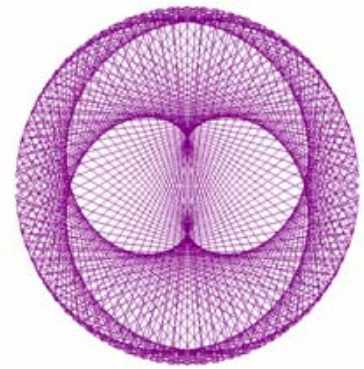
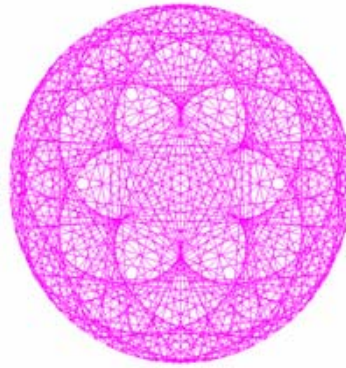
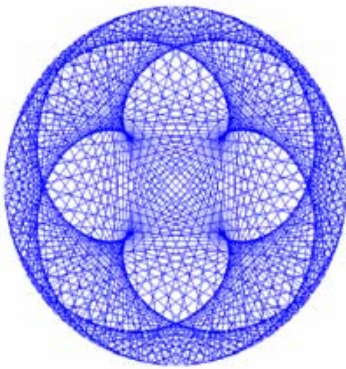
In the Pickover's book the story of Mygalomorph goes like this:

*"Consider a race of spider-beings named Mygalomorphs who spend their days spinning webs upon circular frames. Status in their society is based on the beauty of their webs. To create the web patterns, the spiders string a straight piece of web from one point on the circle to another. Usually the patterns are dull and uninspiring, and therefore most spiders are relegated to lower societal classes.*

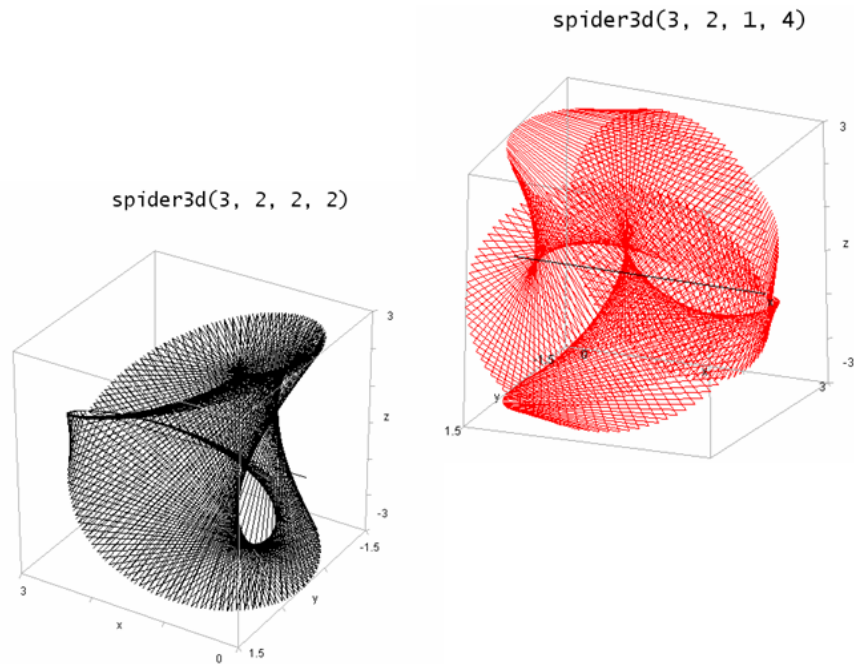
*One day, a rather intelligent Mygalomorph let a straight web piece amble around the circumference of the circle, the front end going six times as fast as the rear. In other words, every time the rear of the straight web moved one space, the front end moved six. After a few moments' contemplation, the Mygalomorph realized that by the time the fast end has completed one trip around the circle, the slow end had traveled just a sixth of the way around. His web grew ever more intricate as he continued weaving. His forelimbs moved back and forth with lightning speed."*

Enjoy the patterns of his spider nets:

```
mygalo2(r, mov1, mov2, i,  $\theta$ , l) :=
  Prog
  i := 1
  l := []
  Loop
  If i > 360 exit
   $\theta := i \cdot \pi / 180$ 
  l := APPEND(l, [[r·COS(mov1· $\theta$ ), r·SIN(mov1· $\theta$ ); r·COS(mov2· $\theta$ ), r·SIN(mov2· $\theta$ )]])
  i := i + 1
  l
```

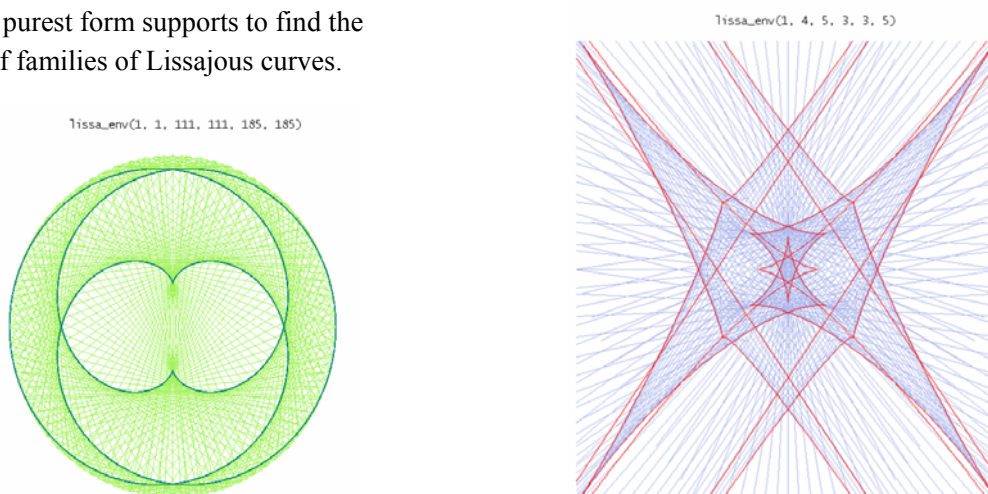


Even in the third dimension:

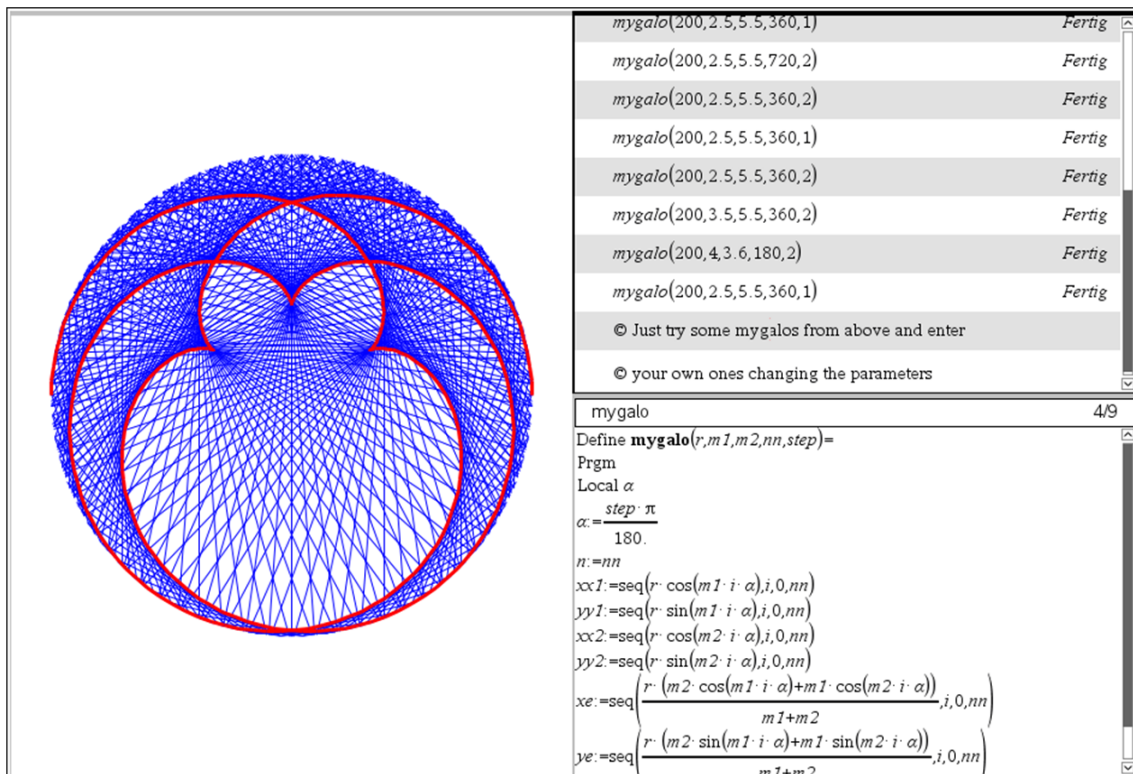


$$\text{lissa\_env}(r1, r2, a, b, c, d) := \left[ - \frac{r1 \cdot r2 \cdot (d \cdot r1 \cdot \cos(a \cdot t) - \cos(d \cdot t) - (b \cdot r1 \cdot \cos(b \cdot t) \cdot \cos(c \cdot t) - c \cdot r1 \cdot \sin(b \cdot t) \cdot \sin(c \cdot t) + r2 \cdot (d \cdot \cos(c \cdot t) \cdot \cos(d \cdot t) + c \cdot \sin(c \cdot t) \cdot \sin(d \cdot t))) - r1 \cdot \cos(a \cdot t) \cdot (b \cdot r1 \cdot \cos(b \cdot t) - d \cdot r2 \cdot \cos(d \cdot t)) + a \cdot r1 \cdot \sin(a \cdot t) \cdot (r1 \cdot \sin(b \cdot t) - r2 \cdot \sin(d \cdot t)) - r2 \cdot (b \cdot r1 \cdot \cos(b \cdot t) \cdot \cos(c \cdot t) \cdot \sin(d \cdot t)) - \cos(c \cdot t) \cdot (a \cdot \sin(a \cdot t) \cdot (r1 \cdot \sin(b \cdot t) - r2 \cdot \sin(d \cdot t)) - b \cdot r2 \cdot \cos(b \cdot t) \cdot \cos(c \cdot t))}{r1 \cdot \cos(a \cdot t) \cdot (b \cdot r1 \cdot \cos(b \cdot t) - d \cdot r2 \cdot \cos(d \cdot t)) + a \cdot r1 \cdot \sin(a \cdot t) \cdot (r1 \cdot \sin(b \cdot t) - r2 \cdot \sin(d \cdot t)) - r2 \cdot (b \cdot r1 \cdot \cos(b \cdot t) \cdot \cos(c \cdot t) \cdot \sin(d \cdot t)) - \cos(c \cdot t) \cdot (a \cdot \sin(a \cdot t) \cdot (r1 \cdot \sin(b \cdot t) - r2 \cdot \sin(d \cdot t)) - b \cdot r2 \cdot \cos(b \cdot t) \cdot \cos(c \cdot t))} \cdot t + c \cdot r1 \cdot \sin(b \cdot t) \cdot \sin(c \cdot t) - r2 \cdot (d \cdot \cos(c \cdot t) \cdot \cos(d \cdot t) + c \cdot \sin(c \cdot t) \cdot \sin(d \cdot t)) \right], \\
 \frac{r1 \cdot r2 \cdot (r1 \cdot \cos(a \cdot t) \cdot (b \cdot r1 \cdot \cos(b \cdot t) \cdot \sin(d \cdot t) - d \cdot \sin(b \cdot t) \cdot \cos(d \cdot t)) + a \cdot \sin(a \cdot t) \cdot \sin(d \cdot t) \cdot (r1 \cdot \sin(b \cdot t) - r2 \cdot \sin(d \cdot t)) - b \cdot r2 \cdot \cos(b \cdot t) \cdot \cos(c \cdot t) \cdot \sin(d \cdot t) - \sin(b \cdot t) \cdot (c \cdot r1 \cdot \sin(b \cdot t) \cdot \sin(c \cdot t) - r2 \cdot (d \cdot \cos(c \cdot t) \cdot \cos(d \cdot t) + c \cdot \sin(c \cdot t) \cdot \sin(d \cdot t)))}{r1 \cdot \cos(a \cdot t) \cdot (b \cdot r1 \cdot \cos(b \cdot t) - d \cdot r2 \cdot \cos(d \cdot t)) + a \cdot r1 \cdot \sin(a \cdot t) \cdot (r1 \cdot \sin(b \cdot t) - r2 \cdot \sin(d \cdot t)) - r2 \cdot (b \cdot r1 \cdot \cos(b \cdot t) \cdot \cos(c \cdot t) \cdot \sin(d \cdot t) - \sin(b \cdot t) \cdot (c \cdot r1 \cdot \sin(b \cdot t) \cdot \sin(c \cdot t) - r2 \cdot (d \cdot \cos(c \cdot t) \cdot \cos(d \cdot t) + c \cdot \sin(c \cdot t) \cdot \sin(d \cdot t)))} \\
 \cdot \cos(c \cdot t) + c \cdot r1 \cdot \sin(b \cdot t) \cdot \sin(c \cdot t) - r2 \cdot (d \cdot \cos(c \cdot t) \cdot \cos(d \cdot t) + c \cdot \sin(c \cdot t) \cdot \sin(d \cdot t)) \right]$$

CAS in its purest form supports to find the envelopes of families of Lissajous curves.



A challenge for TI-Nspire – and for me, of course, too:



The image shows a TI-Nspire interface. On the left is a graph of a mygalos curve, which is a complex, multi-lobed shape formed by the intersection of many circles. The curve is plotted in blue, with several red circles overlaid to show its construction. On the right is a list of commands, each followed by the word "Fertig" (German for "finished"). The commands are:

- mygalo(200,2.5,5.5,360,1)
- mygalo(200,2.5,5.5,720,2)
- mygalo(200,2.5,5.5,360,2)
- mygalo(200,2.5,5.5,360,1)
- mygalo(200,2.5,5.5,360,2)
- mygalo(200,3.5,5.5,360,2)
- mygalo(200,4.3,6,180,2)
- mygalo(200,2.5,5.5,360,1)

Below the list, there are two instructions:

- © Just try some mygalos from above and enter
- © your own ones changing the parameters

At the bottom right, there is a small window titled "mygalo" with the number "4/9". It contains the following code:

```
Define mygalo(r,m1,m2,nn,step)=
Prgm
Local α
α = (step * π) / 180.
n := nn
xx1 := seq(r * cos(m1 * i * α), i, 0, nn)
yy1 := seq(r * sin(m1 * i * α), i, 0, nn)
xx2 := seq(r * cos(m2 * i * α), i, 0, nn)
yy2 := seq(r * sin(m2 * i * α), i, 0, nn)
xe := seq( (r * (m2 * cos(m1 * i * α) + m1 * cos(m2 * i * α))) / (m1 + m2), i, 0, nn)
ye := seq( (r * (m2 * sin(m1 * i * α) + m1 * sin(m2 * i * α))) / (m1 + m2), i, 0, nn)
```

(r\*cos(m2\*θ), r\*sin(m2\*θ))  
with θ = t\*π/180.

$$f(x,y,t) := y - r \cdot \sin\left(\frac{m1 \cdot t \cdot \pi}{180}\right) = \frac{r \cdot \sin\left(m2 \cdot \frac{t \cdot \pi}{180}\right) - r \cdot \sin\left(m1 \cdot \frac{t \cdot \pi}{180}\right)}{r \cdot \cos\left(m2 \cdot \frac{t \cdot \pi}{180}\right) - r \cdot \cos\left(m1 \cdot \frac{t \cdot \pi}{180}\right)} \cdot \left(x - r \cdot \cos\left(m1 \cdot \frac{t \cdot \pi}{180}\right)\right) \quad \blacktriangleright \text{Fertig}$$

$$f1(x,y,t) := \frac{d}{dt}(f(x,y,t)) \quad \blacktriangleright \text{Fertig}$$

solve(f(x,y,t) and f1(x,y,t), {x,y})

$$\blacktriangleright \cos\left(\frac{m2 \cdot \pi \cdot t}{180}\right) - \cos\left(\frac{m1 \cdot \pi \cdot t}{180}\right) \neq 0 \text{ and } x = \frac{r \cdot \left(m2 \cdot \cos\left(\frac{m1 \cdot \pi \cdot t}{180}\right) + m1 \cdot \cos\left(\frac{m2 \cdot \pi \cdot t}{180}\right)\right)}{m1 + m2}$$

$$\text{and } y = \frac{r \cdot \left(m2 \cdot \sin\left(\frac{m1 \cdot \pi \cdot t}{180}\right) + m1 \cdot \sin\left(\frac{m2 \cdot \pi \cdot t}{180}\right)\right)}{m1 + m2} \quad \triangleleft$$

This is the parameter representation of the envelope.

We use the derived equation to accomplish the program.

The envelope is presented in the Graph-Application

An inspiring communication with Alfred Roulier from Switzerland helped to overcome plotting deficiencies of TI-NspireCAS – because it is not possible to program plots as it was possible with Voyage 200 and TI-92. Alfred inspired me to work with LUA (a script language enabling programming the TI-Nspire).

```
Anzahl Schritte/number of steps  n:=360      Radien  r1:=200 ▶ 200  r2:=250 ▶ 250
Parameter  a:=1 ▶ 1  b:=1 ▶ 1  c:=6 ▶ 6  d:=6 ▶ 6
Streckenendpunkte / endpoints of the segments
x1:=seq(r1·cos(a·i·π/180.),i,1,n)
▶ { 200.,200.,200.,200.,199.,199.,199.,198.,198.,197.,196.,196.,195.,194.,193.,192.,191.,190.,189
y1:=seq(r1·sin(b·i·π/180.),i,1,n)
▶ { 3.49,6.98,10.5,14.,17.4,20.9,24.4,27.8,31.3,34.7,38.2,41.6,45.,48.4,51.8,55.1,58.5,61.8,65.1,6
x2:=seq(r2·cos(c·i·π/180.),i,1,n)
▶ { 249.,245.,238.,228.,217.,202.,186.,167.,147.,125.,102.,77.3,52.,26.1,0.,-26.1,-52.,-77.3,-102.,▶
y2:=seq(r2·sin(d·i·π/180.),i,1,n)
▶ { 26.1,52.,77.3,102.,125.,147.,167.,186.,202.,217.,228.,238.,245.,249.,250.,249.,245.,238.,228.
Farbgang / Colour Changing
h1:=60 ▶ 60  h2:=350 ▶ 350  s:=1 ▶ 1  v:=0.9 ▶ 0.9|farbgang(h1,h2,s,v) ▶ Fertig
red
▶ { 226.,223.,220.,217.,214.,211.,208.,205.,202.,199.,196.,193.,189.,186.,183.,180.,177.,174.,171
```

Script Editor

Skript anhalten Skript fortsetzen Skript festlegen Fokus skript Step Into Step Over Haltepunkte aktivieren Haltepunkte deaktivieren Berechtigungen festlegen

Cahiers T<sup>3</sup> Europe  
Vlaanderen nr. 35

**T<sup>3</sup> EUROPE**

### Aan de slag met LUA

LUA-scripts in TI-Nspire

An-Sofie Bruggeman

Gregory Deroo

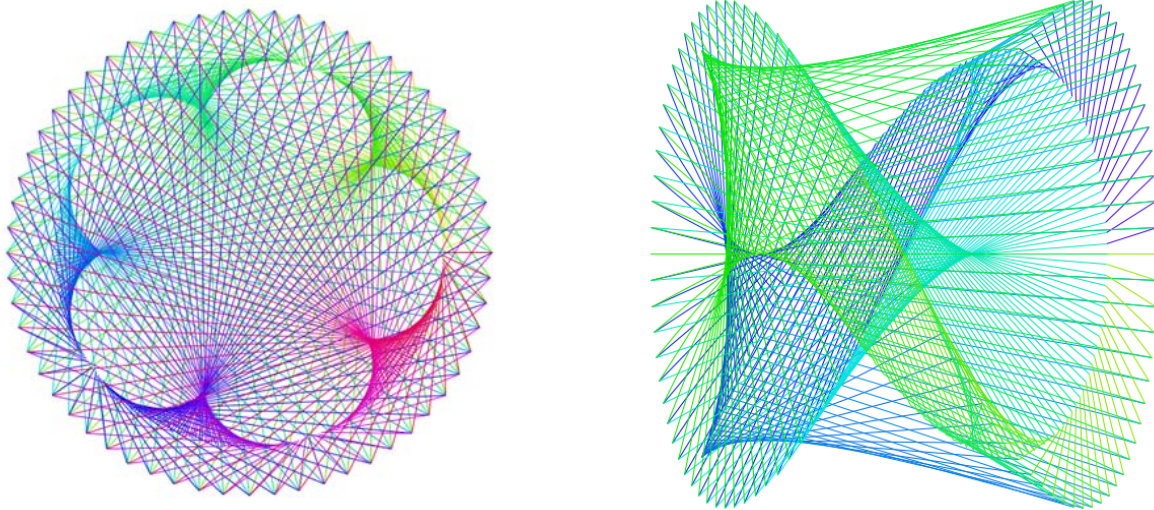
Jan-Klaas D'hulster

Joline Strubbe

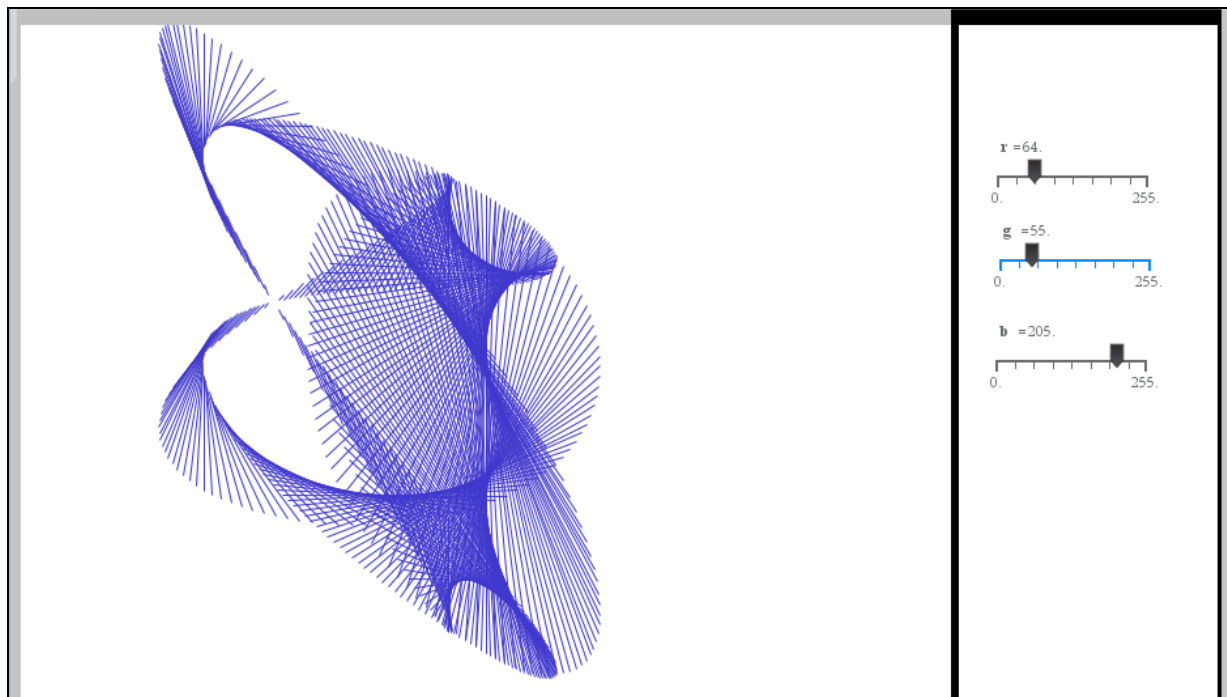
Virginie Vileyn

```
1 platform.apilevel = '1.0'
2
3 function on.create()
4   h=platform.window:height()
5   w=platform.window:width()
6 end
7
8 function on.resize(width,height)
9   h=height
10  w=width
11 end
12
13 function on.enterKey()
14   platform.window:invalidate()
15 end
16
17 function on.paint(gc)
18   --Iterationstiefe übernehmen
19   n=(var.recall("n") or 1)
20   --Koordinatenlisten definieren
21   x1 = {}; x2 = {}; y1 = {}; y2 = {}; rot = {}; gruen = {}; blau = {}
22   for i=1,n do
23     x1[i] = 0 ; x2[i] = 0 ; y1[i] = 0 ; y2[i] = 0 ; rot[i] = 0 ; gruen[i] = 0 ; blau[i] = 0
24   end
25   --Koordinatenlisten übernehmen
26   x1=(var.recall("x1" or 2)) ; x2=(var.recall("x2" or 2))
27   y1=(var.recall("y1" or 2)) ; y2=(var.recall("y2" or 2))
28   --Farblisten übernehmen
29   rot=(var.recall("red" or 1)) ; gruen=(var.recall("green" or 1)) ; blau=(var.recall("blue" or 1))
30   --Linien zeichnen
31   for i=1,n do
32     gc:setColorRGB(rot[i],gruen[i],blau[i])
33     gc:drawLine(w/2+x1[i],h/2-y1[i],w/2+x2[i],h/2-y2[i])
34   end
35 end
```

There is a great booklet on LUA (in Dutch) produced by the Flemish section of T<sup>3</sup>.



This are two products of a LUA script performing continuous change of colours.



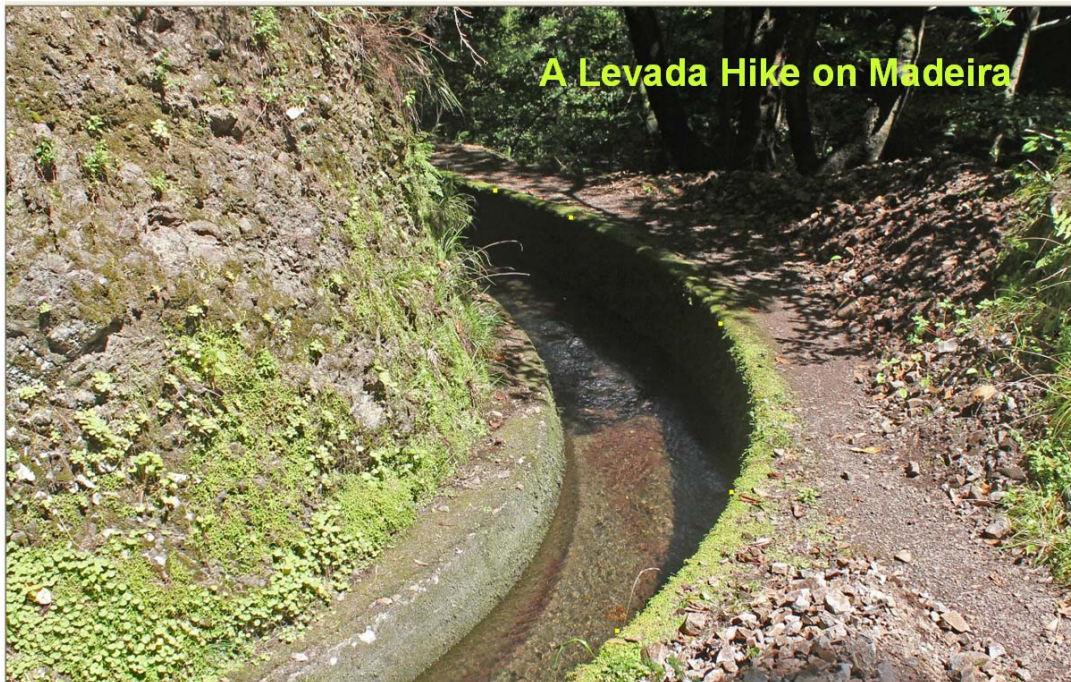
Another LUA-product showing sliders for changing the RGB colour mix.

I'd like to recommend Steve Arnold's great tutorial for LUA scripts:

[http://compasstech.com.au/TNS\\_Authoring/Scripting/index.html](http://compasstech.com.au/TNS_Authoring/Scripting/index.html)

Talking about "Inspirations from Outside" I have to mention our holidays and travels. Some of you know that the background pictures imported to *DERIVE* and TI-Nspire have always been a rich resource for investigation and modelling.

Last spring we were on a hiking week on the wonderful island of Madeira.



I mark five points on the levada and try to find the respective equation, assuming that it might be a conic.

The first calculation gives the equation of the curve in implicit form. Its form is convincing me that the mapping of the curve is a conic, so it must be a conic in reality, too.

The next calculation gives details about the conic.



$$\begin{bmatrix} -0.13 & 0.74 \\ 0.075 & 0.612 \\ 0.48 & 0.11 \\ 0.51 & -0.69 \\ 0.2 & -1.35 \end{bmatrix}$$

$$\text{con}(x, y) := x^2 + a \cdot x \cdot y + b \cdot y^2 + c \cdot x + d \cdot y + e = 1$$

SOLUTIONS([con(-0.13, 0.74), con(0.075, 0.612), con(0.48, 0.11), con(0.51, -0.69), con(0.2, -1.35)], [a, b, c, d, e])

[[-1.410029771, 2.445638703, 4.557897838, 2.339815427, -1.630713315]]

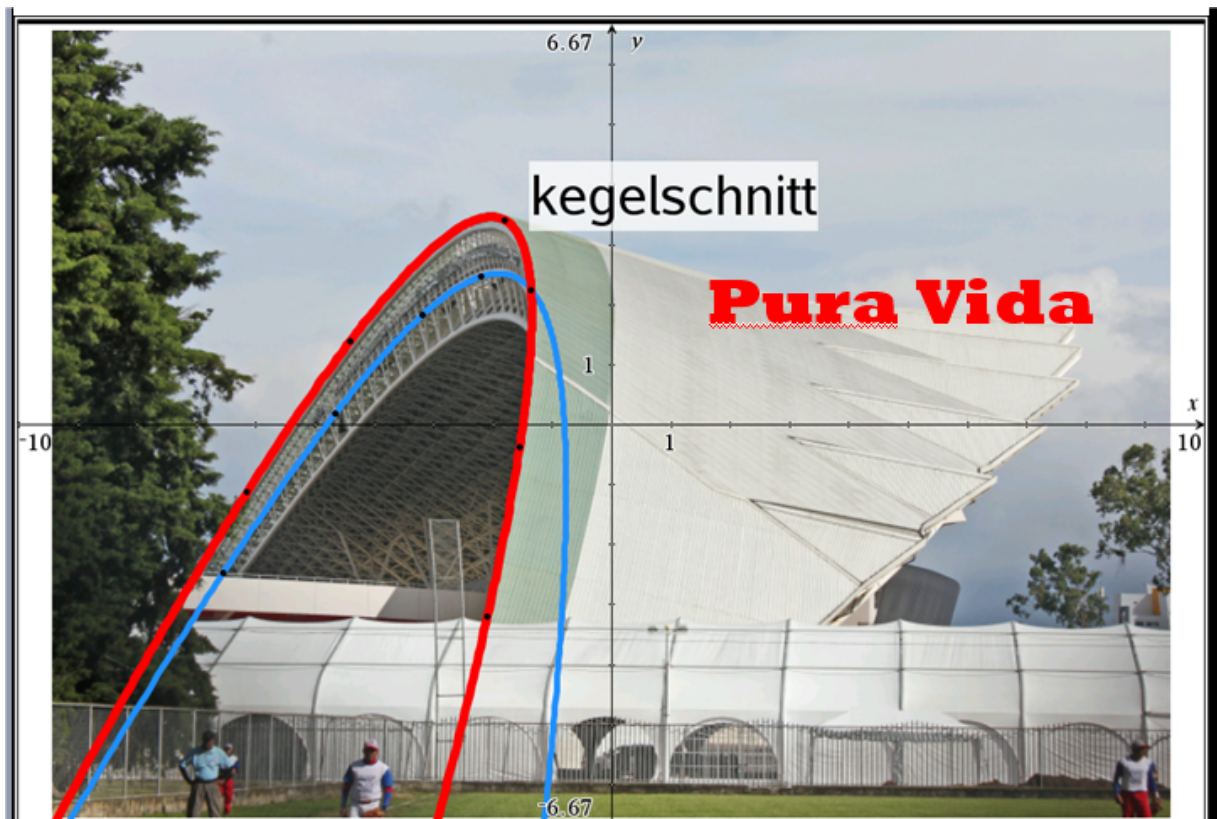
$$x^2 + -1.410029771 \cdot x \cdot y + 2.445638703 \cdot y^2 + 4.557897838 \cdot x + 2.339815427 \cdot y - 1.630713315 = 1$$



$$\text{ch\_conic}(x^2 - 1.41 \cdot x \cdot y + 2.45 \cdot y^2 + 4.56 \cdot x + 2.34 \cdot y - 2.63)$$

Typ	Mittelpkt	Hauptachse	Nebenachse	Scheitel
Type	Center	Major axis	Minor axis	Vertices
Ellipse	$[-3.282607304, -1.422138020]$	4.149459228	8.124577271	$\begin{bmatrix} -7.046451201 & -2.950427285 \\ 0.4812365927 & 0.1061512452 \\ -2.502065279 & -3.344443177 \\ -4.063149329 & 0.5001671373 \end{bmatrix}$

Another conic was discovered in Costa Rica in last fall:



This is the football stadium in San José where Costa Rica gained participation at the World Cup 2014 which is held in these days in Brasil, lucky Costa Rica.

Costa Rica is more famous for its coffee beans, its way of life and its wild life: apes, sloths, crocodiles, iguanas, frogs butterflies and birds. We expected to meet parrots and toekans.

Can you become inspired by a bird?



xxv:={-5.58,-4.98,-1.47,-3.4,-0.21}	{-5.58,-4.98,-1.47,-3.4,-0.21}																
yyv:={-0.14,0.69,1.89,1.51,2.06}	{-0.14,0.69,1.89,1.51,2.06}																
LnReg xxv,yyv,1: CopyVar stat.RegEqn,f5: stat.results	"Fehler: Bereichsfehler"																
xxv:={-5.58+6,-4.98+6,-1.47+6,-3.4+6,-0.21+6}	{0.42,1.02,4.53,2.6,5.79}																
LnReg xxv,yyv,1: CopyVar stat.RegEqn,f3: stat.results																	
	<table border="1"> <tr> <td>"Titel"</td> <td>"Logarithmische Regression"</td> </tr> <tr> <td>"RegEqn"</td> <td>"a+b·ln(x)"</td> </tr> <tr> <td>"a"</td> <td>0.634963</td> </tr> <tr> <td>"b"</td> <td>0.840138</td> </tr> <tr> <td>"r<sup>2</sup>"</td> <td>0.996566</td> </tr> <tr> <td>"r"</td> <td>0.998281</td> </tr> <tr> <td>"Resid"</td> <td>"{...}"</td> </tr> <tr> <td>"ResidTrans"</td> <td>"{...}"</td> </tr> </table>	"Titel"	"Logarithmische Regression"	"RegEqn"	"a+b·ln(x)"	"a"	0.634963	"b"	0.840138	"r <sup>2</sup> "	0.996566	"r"	0.998281	"Resid"	"{...}"	"ResidTrans"	"{...}"
"Titel"	"Logarithmische Regression"																
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"ResidTrans"	"{...}"																
$\beta(x)$	$0.840138 \cdot \ln(x) + 0.634963$																
$\beta(x+6)$	$0.840138 \cdot \ln(x+6) + 0.634963$																



YES, I can!!

This is my last slide:

## Conclusion

My examples shall demonstrate that one can find interesting and challenging problems for applying a CAS not only in textbooks.

These problems can be for students of various mathematical levels and just for one's own entertainment.