The Fibonacci Sequence and Systems of Linear Equations

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On page 1.3:

The first 80 terms of the Fibonacci sequence are displayed in column A.

The first 80 terms of the Lucas sequence are displayed in column B.

The first 80 terms of the so−called 'David' sequence are displayed in column C.

Explore systems of two equations in two unknowns whose coefficients and constants are either Fibonacci, Lucas or 'David' numbers.

	A fib	B lucas	C david	D	E	F	G	H
=	=seqgen(u	=seqgen(u	=seqgen(u					
1	1	2	-3					
2	1	1	5					
3	2	3	2					
4	3	4	7					
5	5	7	9					
6	8	11	16					
7	13	18	25					
8	21	29	41					
9	34	47	66					
10	55	76	107					
11	89	123	173					
12	144	199	280					
13	233	322	453					
14	377	521	733					

A1 =1

Pages 1.6–1.8 contain Math Box templates for solving a system of two equations in two unknowns in each of the Fibonacci, Lucas and 'David' sequence cases.

For example:

$$\text{linSolve}\left(\begin{cases}\mathbf{fib}[1]\cdot x+\mathbf{fib}[2]\cdot y=\mathbf{fib}[3]\\ \mathbf{fib}[4]\cdot x+\mathbf{fib}[5]\cdot y=\mathbf{fib}[6]\end{cases},\{x,y\}\right)$$

corresponds to:

$$\text{linSolve}\left(\begin{cases}x+y=2\\ 3\cdot x+5\cdot y=8\end{cases},\{x,y\}\right)$$

On pages 1.6–1.8, click on the slider 'buttons' to systematically vary the parameters t and k.

Fibonacci Sequence:

$$\text{linSolve}\left(\begin{cases}\mathbf{fib}[\mathbf{t}]\cdot x+\mathbf{fib}[\mathbf{t}+\mathbf{k}]\cdot y=\mathbf{fib}[\mathbf{t}+2\cdot \mathbf{k}]\\ \mathbf{fib}[\mathbf{t}+3\cdot \mathbf{k}]\cdot x+\mathbf{fib}[\mathbf{t}+4\cdot \mathbf{k}]\cdot y=\mathbf{fib}[\mathbf{t}+5\cdot \mathbf{k}]\end{cases},\{x,y\}\right)$$

▸ $\{1,1\}$

k =1. t =1.

Lucas Sequence:

$$\text{linSolve}\left(\begin{cases}\textbf{lucas}[t]\cdot x+\textbf{lucas}[t+\textbf{k}]\cdot y=\textbf{lucas}[t+2\cdot \textbf{k}]\\ \textbf{lucas}[t+3\cdot \textbf{k}]\cdot x+\textbf{lucas}[t+4\cdot \textbf{k}]\cdot y=\textbf{lucas}[t+5\cdot \textbf{k}]\end{cases},\{x,y\}\right)$$

▸ $\{1,1\}$

k =1. t =1.

'David' Sequence:

$$\text{linSolve}\left(\begin{cases}\textbf{david}[\textbf{t}]\cdot x+\textbf{david}[\textbf{t}+\textbf{k}]\cdot y=\textbf{david}[\textbf{t}+2\cdot \textbf{k}]\\ \textbf{david}[\textbf{t}+3\cdot \textbf{k}]\cdot x+\textbf{david}[\textbf{t}+4\cdot \textbf{k}]\cdot y=\textbf{david}[\textbf{t}+5\cdot\end{cases}\{x,y\}\right)$$

▸ $\{1,1\}$

k =1. t =1.

What do you notice about the solution for x in each of the three cases?

Student: Type response here.

What do you notice about the solution for y in each of the three cases?

Student: Type response here.

Problem 2

	A fib	B g	C	D	E	F	G	H	I
=	=seqgen(u	=seqgen(u							
1	1	1							
2	1	3							
3	2	4							
4	3	7							
5	5	11							
6	8	18							
7	13	29							
8	21	47							
9	34	76							
10	55	123							
11	89	199							
12	144	322							
13	233	521							
14	377	843							

A1 =1

$$\text{linSolve}\left(\begin{cases} fib[t]\cdot x+fib[t+k]\cdot y=fib[t+2\cdot k] \\ fib[t+3\cdot k]\cdot x+fib[t+4\cdot k]\cdot y=fib[t+5\cdot k] \end{cases},\{x,y\}\right)\to luc(t,k)$$ *Done*

$\text{seq}(luc(1,k),k,1,15)$

$$\begin{bmatrix} 1 & 1 \\ -1 & 3 \\ 1 & 4 \\ -1 & 7 \\ 1 & 11 \\ -1 & 18 \\ 1 & 29 \\ -1 & 47 \\ 1 & 76 \\ -1 & 123 \\ 1 & 199 \\ -1 & 322 \\ 1 & 521 \\ -1 & 843 \\ 1 & 1364 \end{bmatrix}$$

General result:

From the results obtained in question **j**, we can define a new Fibonacci sequence with general term G_k where $G_k = G_{k-1} + G_{k-2}$, $G_1 = 1$ and $G_2 = 3$.

Hence when each constant in the 2×2 system of linear equations is every *k*th Fibonacci number, the general solution is $x = (-1)^{k+1}$ and $y = G_k$.

Linking the Solution and Fibonacci Sequences (1):

$$\frac{\mathbf{fib}[2\cdot \mathbf{k}]}{\mathbf{fib}[\mathbf{k}]} = 1 \qquad \mathbf{g}[\mathbf{k}] = 1$$

$$\text{seq}\left(\frac{\mathbf{fib}[2\cdot \mathbf{k}]}{\mathbf{fib}[\mathbf{k}]},\mathbf{k},1,15\right)$$

$$\text{seq}(\mathbf{g}[\mathbf{k}],\mathbf{k},1,15)$$

$$\text{seq}\left(\frac{\mathbf{fib}[2\cdot \mathbf{k}]}{\ldots},\mathbf{k},1,15\right)=\text{seq}(\mathbf{g}[\mathbf{k}],\mathbf{k},1,15)$$

k =1.

General result (1):

$$fib[2\cdot k]=g[k]\cdot fib[k]$$

So which solution sequence number is a factor of **fib**[2**k**] ?

Linking the Solution and Fibonacci Sequences (2):

$$fib[2^2]=g[2^1]$$

$$fib[2^3]=g[2^1]\cdot g[2^2]$$

$$fib[2^4]=$$

$$fib[2^5]=$$

$$fib[2^6]=$$

General result (2):

$$\mathbf{fib}\left[2^{\mathbf{k}}\right]=\mathbf{g}\left[2^{1}\right]\cdot\mathbf{g}\left[2^{2}\right]\cdot\mathbf{g}\left[2^{3}\right]\cdot\ldots\cdot\mathbf{g}\left[2^{\mathbf{k}-1}\right]$$

Problem 3

	A fib	B g	C	D	E	F	G	H	I
=	=seqgen(u	=seqgen(u							
1	1	1							
2	1	3							
3	2	4							
4	3	7							
5	5	11							
6	8	18							
7	13	29							
8	21	47							
9	34	76							
10	55	123							
11	89	199							
12	144	322							
13	233	521							
14	377	843							

A1 =1

Linking the Solution and Fibonacci Sequences (3):

$\mathrm{fib}[k-t]+\mathrm{fib}[k+t] = 47$

Systematically vary the values of *k* and *t* and study the behaviour of the expressions below for odd and even *t*.

$\mathrm{fib}[k]\cdot g[t] = 21$ $\mathrm{fib}[t]\cdot g[k] = 47$

k =8. t =1.

Problem 4

Arithmetic Sequences and Systems of Linear Equations:

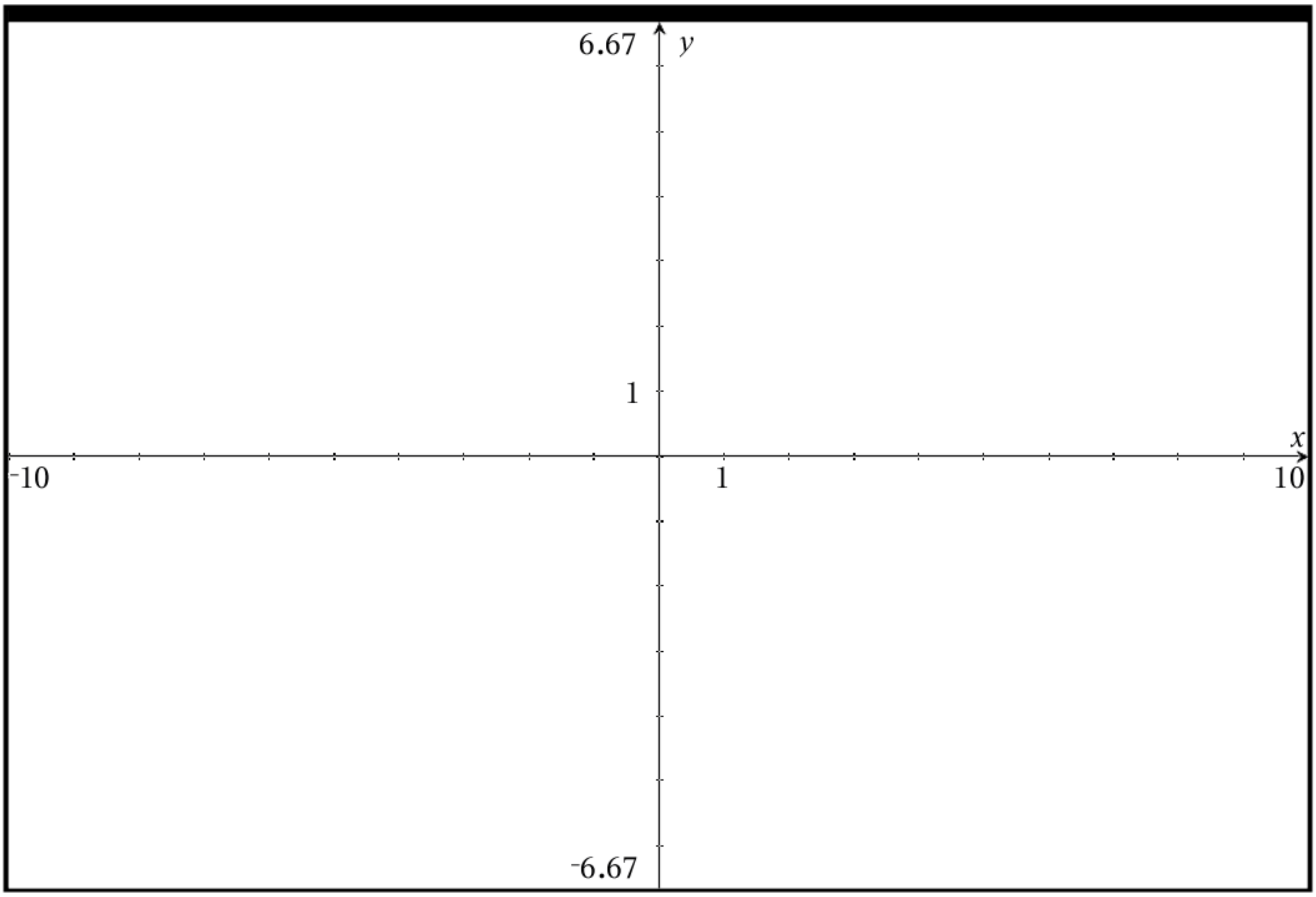
There is a lot more to this than meets the eye.

For example, graph the equations.

Consider the system of equations:

$$\mathbf{m}\cdot x+(\mathbf{m}+\mathbf{p})\cdot y=\mathbf{m}+2\mathbf{p}$$

$$\mathbf{n}\cdot x+(\mathbf{n}+\mathbf{q})\cdot y=\mathbf{n}+2\mathbf{q}$$



$$\text{linSolve}\left(\begin{cases}\mathbf{m}\cdot x+(\mathbf{m}+\mathbf{p})\cdot y=\mathbf{m}+2\cdot\mathbf{p}\\ \mathbf{n}\cdot x+(\mathbf{n}+\mathbf{q})\cdot y=\mathbf{n}+2\cdot\mathbf{q}\end{cases},\{x,y\}\right) \blacktriangleright \{{}^{-}1.,2.\}$$

$$\det\left(\begin{bmatrix}\mathbf{m} & \mathbf{m}+\mathbf{p}\\ \mathbf{n} & \mathbf{n}+\mathbf{q}\end{bmatrix}\right) \blacktriangleright {}^{-}2.$$

m =4.

p =3.

n =2.

q =1.

A Pedagogical Approach to Solving the Arithmetic Sequence Case

$m\cdot x+(m+p)\cdot y=m+2\cdot p \rightarrow e1$ $\quad m\cdot x+(m+p)\cdot y=m+2\cdot p$

$n\cdot x+(n+q)\cdot y=n+2\cdot q \rightarrow e2$ $\quad n\cdot x+(n+q)\cdot y=n+2\cdot q$

Problem 6

Geometric Sequences and Systems of Linear Equations:

Consider the system of equations:

$a\cdot x+a\cdot r\cdot y=a\cdot r^{2}$ and $b\cdot x+b\cdot s\cdot y=b\cdot s^{2}$

$$\text{linSolve}\left(\begin{cases} x+r\cdot y=r^{2} \\ x+s\cdot y=s^{2} \end{cases},\{x,y\}\right)$$

$$y=r+s\,|\,s=\frac{^{-}x}{r}$$

$\text{solve}\left(x+r\cdot y=r^{2},r\right)$

Problem 7