Differences between Expected Answers and the Answers Given by Computer Algebra Systems to School Equations

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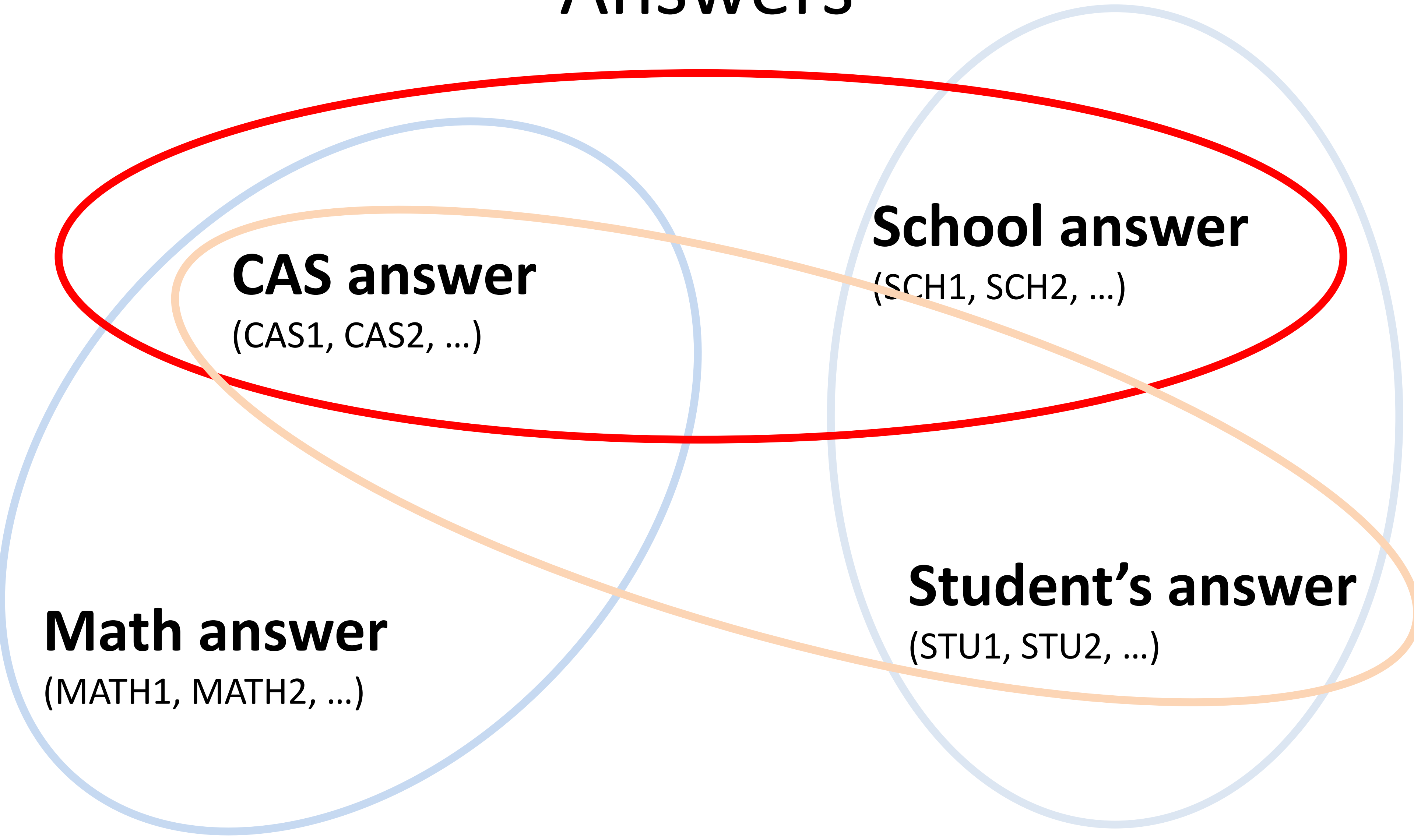
Background

- Computer Algebra Systems
  - Maple, Mathematica, Maxima, Axiom, Wiris …
  - are capable of solving many problems
    - more sophisticated
    - better
- Different users
  - professional use
  - educational purposes
    - different ways
- Different needs, different expectations
  - solvable problems
  - user interface
  - form of answers

Answers


CAS answer
(CAS1, CAS2, …)
School answer
(SCH1, SCH2, …)
Math answer
(MATH1, MATH2, …)
Student’s answer
(STU1, STU2, …)

Michael Wester. *Computer Algebra Systems. A Practical Guide*. 1999

– 542 problems

#	PROBLEM	Ax	De	Mc	Mp	Mm	Mu	Re
L6	derivative of above is 0 & above at 0 is 0	#	#	$\star^{17}$	$\star$	○	#	●
L7	$\log\frac{2\sqrt{r}+1}{\sqrt{4r+4\sqrt{r}+1}} \Rightarrow 0$	[illegible]	○	$\star$	●	○	●	●
L8	[illegible]	●	○			○	●	
L9	$2^{1-z}\Gamma(z)\zeta(z)\cos\frac{z\pi}{2} - \pi^z\zeta(1-z) \Rightarrow 0$	—				●		
	M. Equations							
M1	[illegible]	●		●	●	○	●	●
M2	solve$(3x^3 - 18x^2 + 33x - 19 = 0, \mathbf{R})$	○	●	$\star$	$\star$	○	$\star$	●
M3	solve$(x^4 + x^3 + x^2 + x + 1 = 0)$	$\natural$	●	●	●	○	●	●
M4	verify a solution of the above	●	●	●	●	●	$\star$	●
M5	solve$(x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23)$							
M6	solve$(x^7 - 1 = 0) \Rightarrow x = \{1, \{e^{\pm 2k\pi i/7}\}_{k=1}^3\}$	$\natural$	$\star$	●	●	$\star$	$\natural$	$\natural$
M7	solve$(x^8 - 8x^7 + \cdots - 140x + 46 = 0)$			●	●	●	●	●
M8	solve$(e^{2x} + 2e^x + 1 = z, x)$							
M9	solve$(e^{2-x^2} = e^{-x}) \Rightarrow x = \{-1, 2\}[+$							
M10	solve$(e^x = x) \Rightarrow x = -W_n(-1) \quad (n$							
M11	solve$(x^x = x) \Rightarrow x = \{-1, 1\}$							

● success! (hurrah!)

\# incompletely simplified, but some useful transformations were performed (groan)

ε a surprising error occurred (ack!)

….

Answers offered by CAS

- could be evaluated from different points of view
  - professional user's
    - mathematically correct
    - somewhat flexible output allowed (nuances do not confuse so much)
      - radians/degrees
    - …
  - student's
    - according to school mathematics
    - nuances could be important
    - …
  - …

=?

```
(%i2) solve(x^2=-1);
(%o2) [x = -%i, x = %i]
```

The unexpected answers

confusing and obstructive

or

opportunities (Paul Drijvers) and

a catalyst for rich mathematical discussion (Robyn Pierce, Kaye Stacey)

Opportunity, catalyst

Drijvers, P. (2002)

The first obstacle is: The difference between the algebraic representations provided by the CAS and those students expect and conceive as ’simple’. ... Recognizing equivalent expressions is a central issue in algebra, and still is when working in a computer algebra environment.

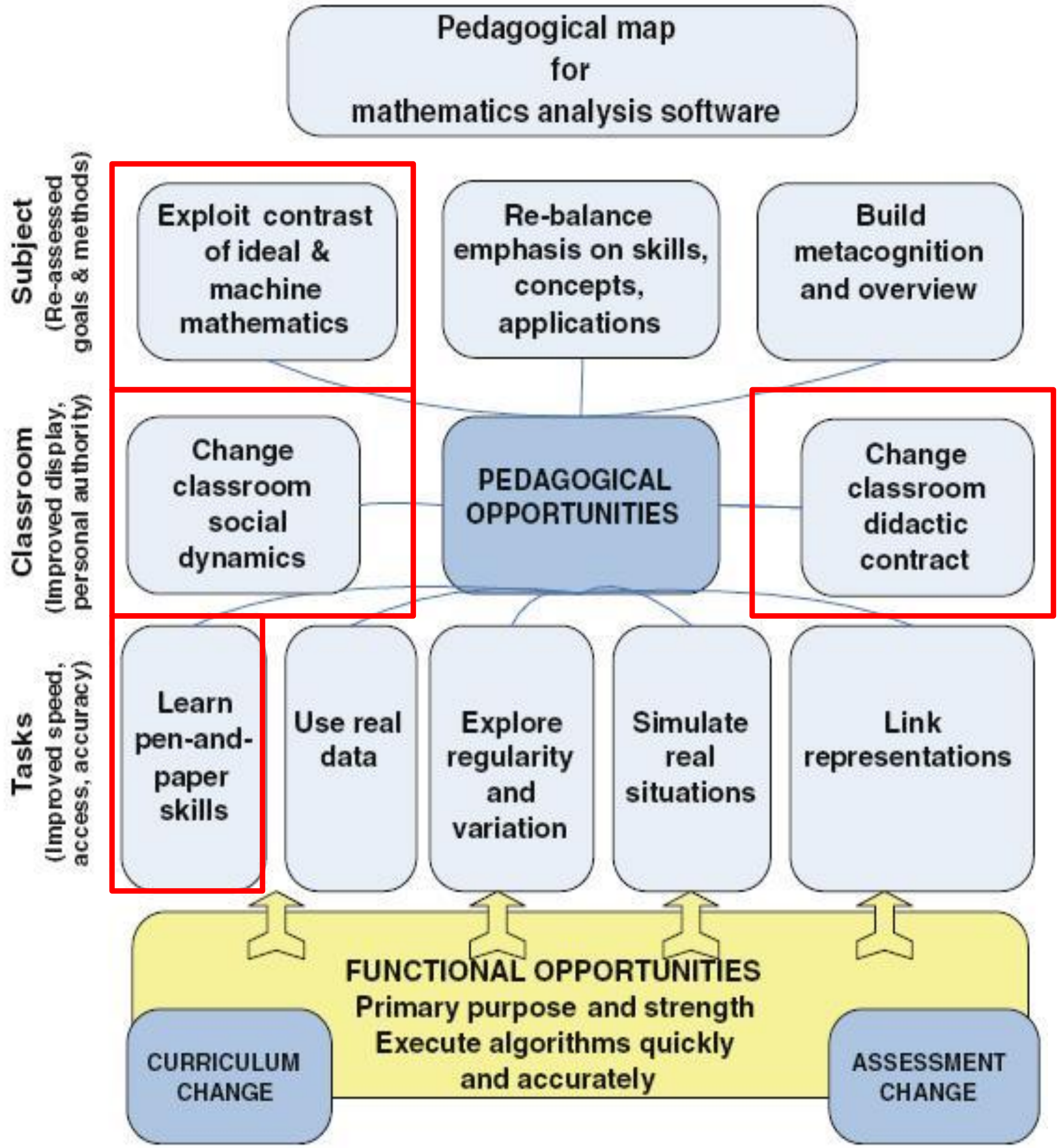
Pierce, R., & Stacey, K. (2010)

Unexpected mathematical results may be distracting and disheartening, but they are also pedagogical opportunities since they be used to provoke rich mathematical discussion.

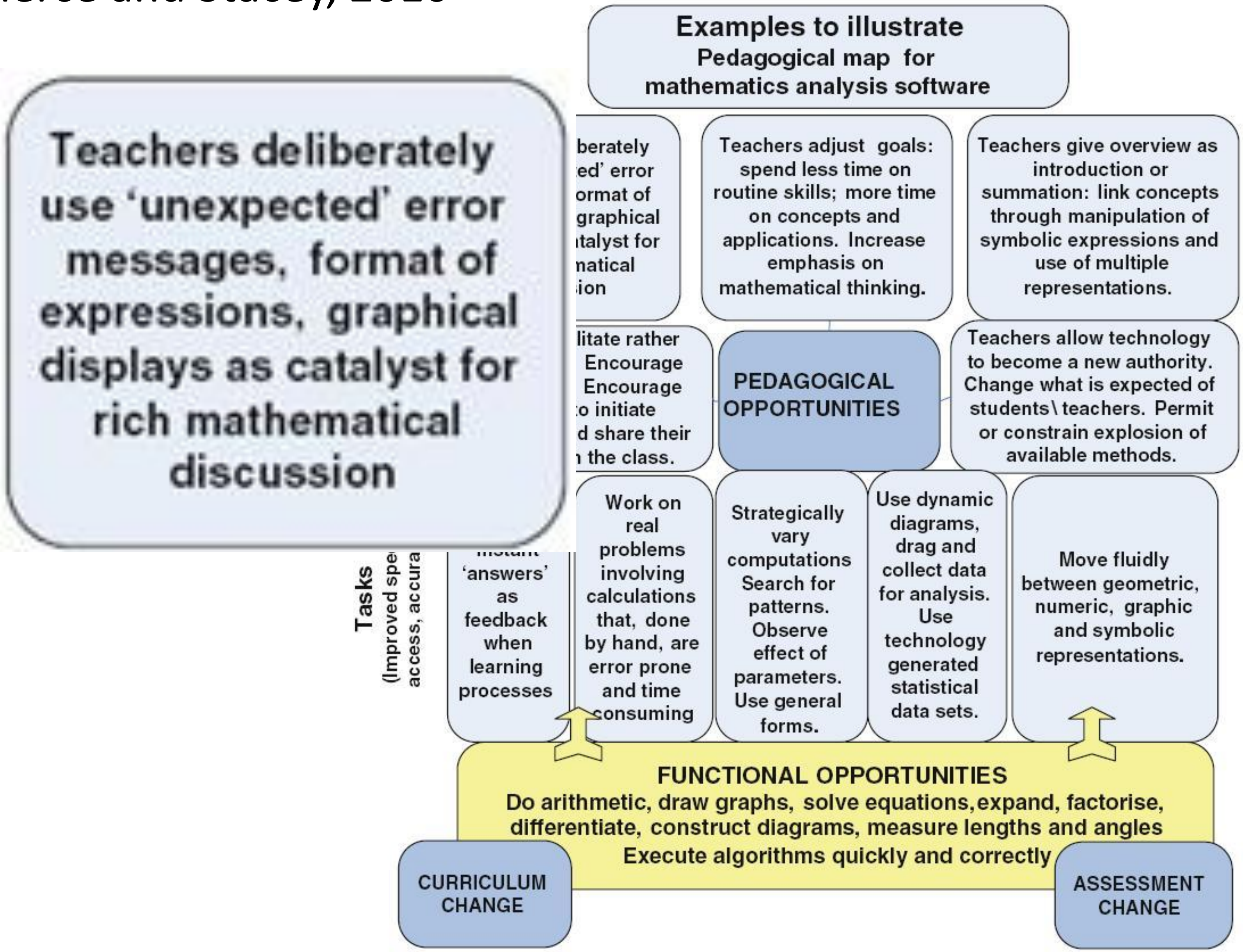
Buteau, C., Marshall, N., Jarvis, D. H., & Lavicza, Z. (2010)

Although practitioners have to deal with unusual or unexpected behaviour of CAS, this was occasionally shown to provide pedagogical opportunities.

Pierce and Stacey, Mapping pedagogical opportunities provided by mathematics analysis software. *International Journal of Computers for Mathematical Learning*, *15*(1), 1-20. 2010



Pierce and Stacey, 2010



Dozens of Equations

- Linear $5x - 11 = 0$
- Quadratic $2x^2 - 4x - 5 = 0$
- Fractional equations $\dfrac{x+1}{x-1} = 0$
- Equations that contain an absolute value of an expression $|x+5| = 3$
- Irrational $\sqrt[3]{5x+7} - \sqrt[3]{5x-12} = 1$
- Exponential $4^x = 64$
- Logarithmic $\log_2 x = 4$
- Trigonometric $\sin x = \dfrac{1}{2}$
- Literal $ax = 1$

- Most answers offered by CAS are customary for school, but there are some answers that would be somewhat incompatible with the teaching practice in school

Related

- Related to operations and functions
  - New operation or function → new equation (logarithm → logarithmic equation)
  - Restrictions
    - division by zero
    - a negatiive number under square root
    - …

- Out of school mathematics
  - Infinity
  - Branch cuts for complex elementary functions
  - …

Questions

- Is the answer offered by a CAS equivalent to the answer required at school?
- Would it be easy for a student, working manually, to transform the answer into the required form?
- How about if the student uses a CAS?
- What issues could be raised for discussion?
  - What differences and relations between CAS and school answers could be identified and explained?
  - Relevant in a school context!?
    - Beyond the regular school level!?

Aim

- The aim is to present a spectrum of possible differences and to highlight the situations (equations, CAS) in which a particular type of discrepancy could occur.
  - Only some in the presentation

Computer Algebra Systems

- GeoGebra 4.4 http://www.geogebra.org/
  - Giac/Xcas
- Maxima http://maxima.sourceforge.net/
- Sage http://www.sagemath.org/
  - Maxima
- Wiris http://www.wiris.com/en/cas
- Wolfram Alpha http://www.wolframalpha.com/
- Xcas http://www-fourier.ujf-grenoble.fr/~parisse/giac.html

Not yet

- Axiom http://www.axiom-developer.org/
- Yacas http://yacas.sourceforge.net/

Command

- **`solve`**
- **`to_poly_solve`**
- **`findroot`**

Different form

- Form
  - Numbers
    - Linear $2\frac{1}{5}$ or $\frac{11}{5}$
    - Quadratic
  - Solution series
    - Trigonometric

$$x = -\frac{1}{2} + (-1)^n \frac{\pi}{12} + \frac{n\pi}{4},\ n \in \mathbb{Z}$$

Results: More digits

$$x = \frac{1}{6}(3\,\pi\,n + \pi - 3) \approx 0.16667\,(9.4248\,n + 0.14159) \text{ and } n \in Z$$

$$x = \frac{1}{12}(6\,\pi\,n + \pi - 6) \approx 0.083333\,(18.850\,n - 2.8584) \text{ and } n \in Z$$

solve	$2x^2 - 4x - 5 = 0$

Results:

$x = \frac{1}{2}(2 - \sqrt{14}) \approx -0.87083$

$x = \frac{1}{2}(2 + \sqrt{14}) \approx 2.8708$

(%i1) solve(2*x^2-4*x-5=0,x);

(%o1) $[x = -\frac{\sqrt{14}-2}{2}, x = \frac{\sqrt{14}+2}{2}]$

```
solve(2*x^2-4*x-5==0,x)
    [x == -1/2*sqrt(14) + 1, x == 1/2*sqrt(14) + 1]
```

$\text{solve}(2x^2-4x-5=0) \rightarrow \left\{\left[x = \frac{\sqrt{14}}{2}+1\right], \left[x = -\frac{\sqrt{14}}{2}+1\right]\right\}$

Different answers

- $\sqrt{x-2} = \sqrt{3+2x}$
- -5
  - GeoGebra
  - Maxima
  - Sage
  - WolframAlpha
  - Xcas
- No solution
  - Wiris

School

- $\sqrt{x-2} = \sqrt{3+2x}$
- $x - 2 = 3 + 2x$
- $-x = 5$
- $x = -5$

$$\sqrt{-7}$$

- $x - 2 \geq 0$ and $3 + 2x \geq 0$
- $x \geq 2$ and $x \geq -3/2$
- No solution???
- No real solution???
  - But -5 is real!?

$$\sqrt{-1}$$

Different answers

- $\frac{\tan^2 x}{\tan x} = 0$

No solution or

$$x = \pi n \approx 3.1416 n \text{ and } n \in \mathbb{Z}$$

Real?

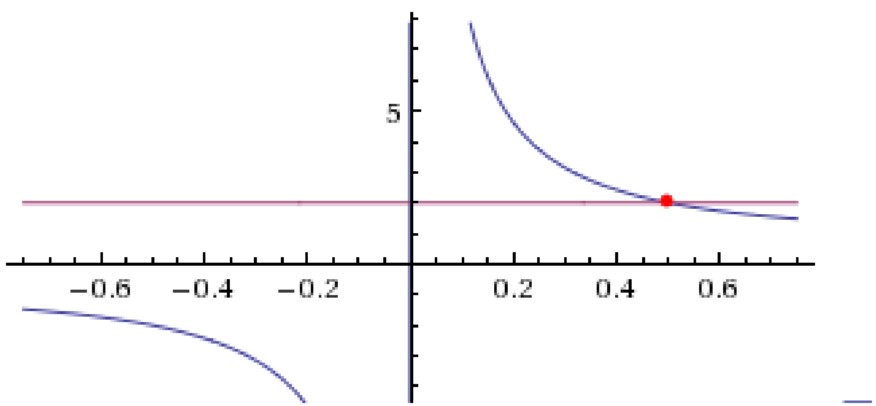
Input interpretation:

solve	$\dfrac{3^x + 3^{-x}}{3^x - 3^{-x}} = 2$

Result:

$$x = \frac{\log(3) + 2\, i\, \pi\, n}{\log(9)} \approx 0.45512\,(1.0986 + 6.2832\, i\, \text{ı}$$

Plot:



solve((3^x+3^(-x))/(3^x-3^(-x))=2) real

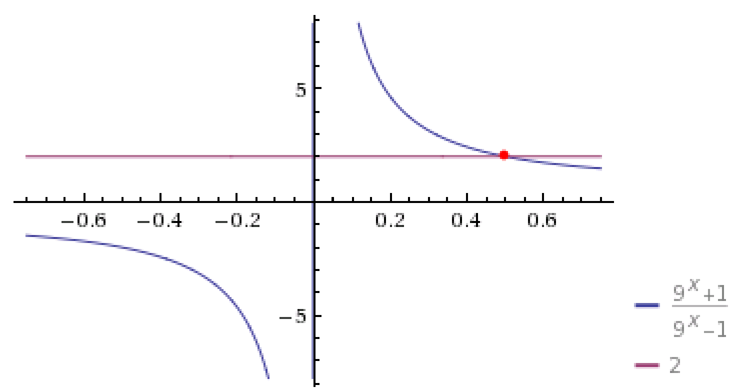
Input interpretation:

solve	$\dfrac{3^x + 3^{-x}}{3^x - 3^{-x}} = 2$	over the reals

Result:

$$x = \frac{1}{2} \approx 0.50000$$

Plot:



Former equation

solve(sqrt(x-2)=sqrt(3+2*x)) real

Input interpretation:

solve	$\sqrt{x-2} = \sqrt{3+2x}$	over the reals

Result:

(no real solutions)

Discussions in lessons

- Whole class
- Small groups
- Pairs
- Role of the teacher

- Discussions in case of trigonometric equations
  - *Students' Comparison of Their Trigonometric Answers with the Answers of a Computer Algebra System in Terms of Equivalence and Correctness*

Lessons

- first-year university students
- course ”Elementary mathematics”
  - a somewhat repetitious course of school mathematics
- 90 minutes
- students in pairs (discussion!!!)
  - discussions were audio-taped
- The students
  - first solved an trigonometric equation (correctly or not) without CAS
  - then with a particular CAS
  - analyzed differences, equivalence and correctness of their own answers and CAS answers

A student is charged with the task of comparing the answers

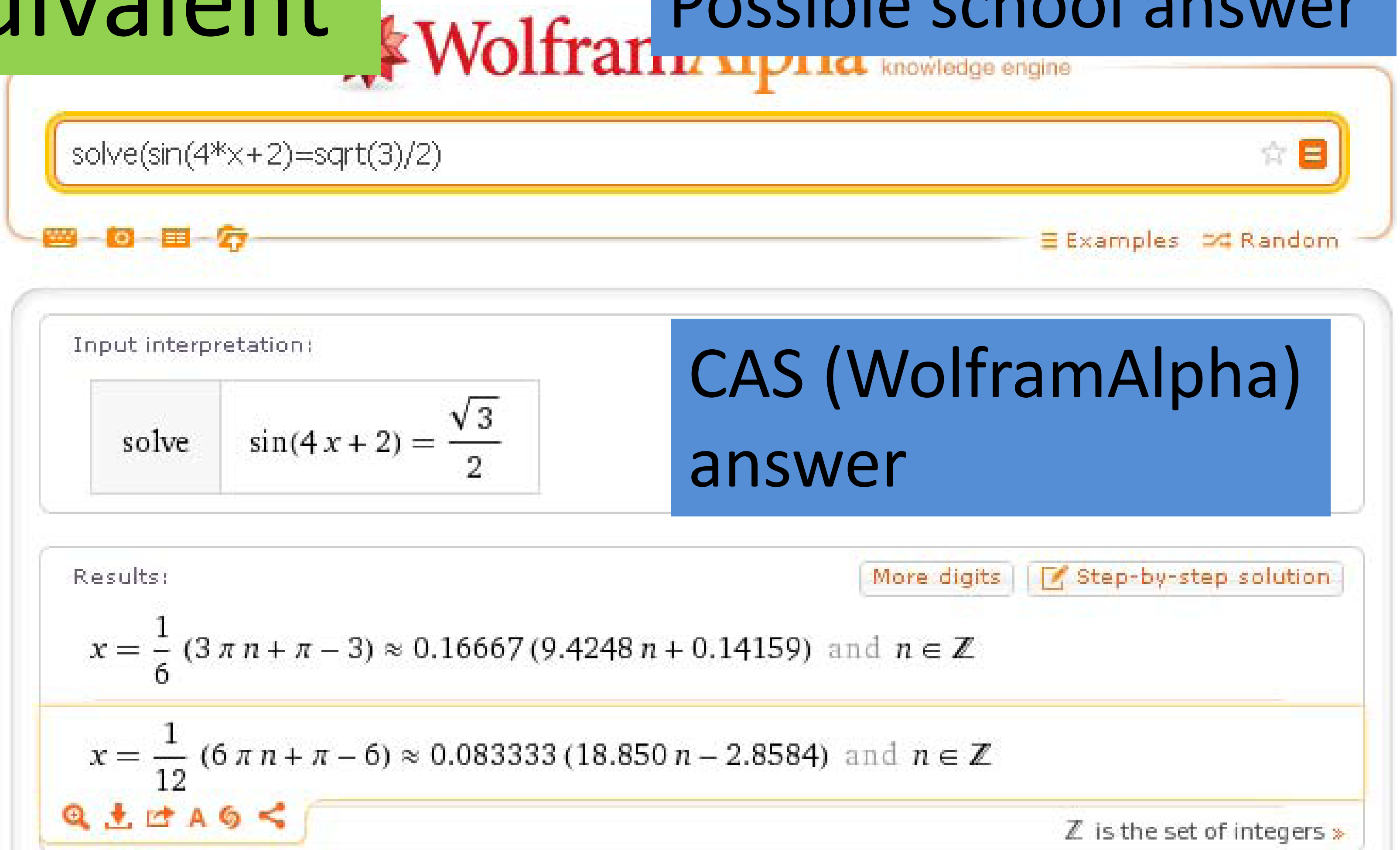
- What will happen when students themselves are encouraged to analyze differences, equivalence and correctness of their own answers and CAS answers?
- What differences do they notice foremost?
- How do they understand correctness of the answers?
- Are students able to ascertain equivalence/non-equivalence?
- How do they explain equivalence/non-equivalence?
- Are there any differences in this regard between different types of equations and answers?

Different Forms of General Solution

$$\sin(4x + 2) = \frac{\sqrt{3}}{2} \qquad x = -\frac{1}{2} + (-1)^n \frac{\pi}{12} + \frac{n\pi}{4}, \ n \in \mathbb{Z}$$

Equivalent

Possible school answer



CAS (WolframAlpha) answer

Order

- The students had worksheets with equations and tasks
- The order of solvable equations
  - prescribed
- The students
  - first solved an equation (correctly or not) without a CAS
  - then with a particular CAS
- WolframAlpha (in the first three equations)
- A specific CAS was prescribed for the equation
  - the expected difference between the students' answers and the CAS answer
  - initiate an ”intrigue”, the effect of different representations

- Solve an equation $\sin(4x+2) = \frac{\sqrt{3}}{2}$ (without the computer at first).
  - How confident are you in the correctness of your answer?
- Solve the equation with the CAS WolframAlpha using the command solve.

  - How unexpected is the CAS answer at first view?
  - Analyze the accordance of your answer with the CAS answer! If you want to complement/correct your solution, please use the green pen.
  - What are the differences between your answer and the CAS answer?
  - How are your answer and the CAS answer related (analyze equivalence/nonequivalence, particular solutions/general solutions)?
  - Rate the correctness of your (possibly corrected) answer.
  - Rate the correctness of the CAS answer.
  - Rate the equivalence/non-equivalence of your (possibly corrected) and CAS answers.

$$\sin(4x + 2) = \frac{\sqrt{3}}{2}$$

$$2 \sin 2x \cos 2x + \cos 2x = 0$$

in the interval [-30°; 0]

$$\frac{\tan^2 x}{\tan x} = 0$$

$$\tan^3 x = \tan x$$

$$\tan(x + \frac{\pi}{4}) = 2 \cot x - 1$$

$$2 \cos^2 x + 4 \cos x = 3 \sin^2 x$$

$$1 - \cos x = \sqrt{3} \sin x$$

Three equations

- 112 instances of equation-solving (38 pairs of students)
- The student worksheets and audio-tapes (in questionnable places) were analyzed
- For each equation in paper
  - how students solved the equation (common mistakes)
  - correctness of the students' answer / the students' confidence in their answer
  - How unexpected is the CAS answer at first view?
  - correctness of the students' answer / the students' opinion about correctness of their answer
  - correctness of the students' answer / the students' opinion about correctness of CAS answer
  - equivalence of the students' and CAS answer / the students' opinion about equivalence of their and CAS answer
  - What are the differences between your answer and the CAS answer?
  - How are your answer and the CAS answer related (analyse equivalence/non-equivalence, particular solutions/general solutions)?

Answers from the experiment

What will happen when students themselves are encouraged to analyse differences, equivalence and correctness of their own answers and CAS answers?

- A very easy answer is that they can work on trigonometry for a whole lesson
- The task seemed to be new for the students
  - Usually, only the solution of an equation is needed and not more
  - The format seemed to be interesting and catching
- These somewhat unexpected answers could support the discussion and provide a possibility to activate students
- The role of the teacher was mainly introduce the lesson and answer some questions during the lesson

Answers from the experiment

Are there any differences in this regard between different types of equations and answers?

- The choice of equation (particularly by answer) is very important
- It would be useful to evaluate the "distance" between the CAS answer and school answer (or probable students' answer)
  - For example, the "distance" seems to be too large in case of the first equation

$$x=(-1)^n\frac{\pi}{12}-\frac{1}{2}+\frac{n\pi}{4}, n\in Z$$

$$x=\frac{1}{6}(3\pi n+\pi-3)\approx 0.16667(9.4248n+0.14159) \text{ and } n\in \mathbb{Z}$$

$$x=\frac{1}{12}(6\pi n+\pi-6)\approx 0.083333(18.850n-2.8584) \text{ and } n\in \mathbb{Z}$$

As a teacher

- the author could argue that the lessons were successful
  - It was also confirmed by the actual teachers of these groups
- task of comparing their own answers and CAS answers was interesting to the students
- they became accustomed to the style of the lesson and actively discussed the topic of trigonometry

As a researcher

- The method seems to be fruitful in research context.
- The paper data and audio-tapes complement each other and give a good overview about the students' activities during the solving process.

Finally

- the method of asking students to compare their own answers with CAS answers
  - seems to have potential in the context of learning as well as research
  - further work is certainly needed
  - could contribute of the usage of computer-based tools for doing mathematics in different ways
- The students see that calculations can be performed faster and easier.
- One should understand that evaluation of a CAS answer may not be so fast and easy.
- The abilities of critical thinking are likely to be developed by the exercises.

Further work

- Prepare worksheets in case of other types of equations
    - $\sqrt{x-2} = \sqrt{3+2x}$???

- Experiments with students

M. Thomas, J. Monaghan, and R. Pierce, Computer Algebra Systems and Algebra: Curriculum, Assessment, Teaching, and Learning, in The Future of the Teaching and Learning of Algebra. The 12th ICMI Study, Kluwer Academic, 2004, pp. 155-186.

> *There is no doubt in the minds of those involved in that forum that in the hands of teachers and learners CAS have the potential to change the teaching and learning of algebra, and possibly to do so in radical ways. However, it is far less clear exactly what these ways are and how such change might best be accomplished.*