

## From Calculus to Dynamical Systems through CAS

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### Abstract

Several factors have influenced in order to use CAS or DGS in the learning-teaching process, such as: development of new technologies, availability to use computers by teachers and students, widespread use of the Internet, etc. Even more, the trend to include CAS and DGS in the curricula of some undergraduate studies claims that those must be used in teaching process. This paper is geared towards exploring the possibilities of the use of CAS and DGS in higher education. Thus, we did a new approach showing four possible uses in the classroom or outside the university in order to encourage professors and students to use them in some topics related with variational thinking, namely, (i) as an acquisition and / or consolidation of the concept of derivative through the graph of the derivative of a function having as data just the graph of the primitive function; (ii) as a support for figuring out some mathematical formulas, taking a particular case: the parametric equations of the cycloid; (iii) as a support for determining the limits of integration in cylindrical and spherical coordinates that belongs to a 3D space; and (iv) as support for understanding of the concept of singular solution and its enveloping for a differential equation. For this purpose, we propose apps made with GeoGebra and Maple, which could be used to improve the understanding of the mathematical topics selected.

The activities developed with CAS or DGS, allow students acquire enough mathematical competencies to understand deeper the concepts involved. This seems to be a good indicator both of the development of variational thinking and of the metacognitive process.

**Keywords:** Variational thinking, Geogebra, Maple, CAS and DGS in higher education, derivative of a function, cycloid, cylindrical and spherical coordinates, singular solution, envelope for a differential equation.

### Introduction

With the advent and accelerated development of technologies, starting with the calculators that today allow us to get graphs, dynamic geometry software (DGS) and computer algebra systems (CAS) for laptops and tablets, all of them have powered the opportunities of educative intervention, allowing the inclusion of the iconic mode in representation of dynamic phenomena as a previous step to the symbolic mode [1].

The inclusion of the new technologies permits a wide experiences gamut that a few years ago was not available nor for professors neither for students. For example, be able to present the variation of the slope of a curve, point to point, the visualization of parametric functions as a cycloid, three-dimensional surfaces projection for the determination of integration limits, and so on, give life to mathematical abstract entities and help to facilitate procedures and to improve understanding of concepts.

Nowadays, the DGS, and the corresponding CAS, are easily accessible through the web and inclusive there are a variety of simulations that allow students to solve almost any task or homework the teacher asks for.

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All of this represents two sides of the same coin: a) it is useful for learning, and b) it facilitates the interaction with students of poor mathematic knowledge.

The visualization on the screen of certain assertions enables student the questioning and/or confirmation by zooming and rotating 3D figures, and develops of variational thinking, that at the end, it allows to extract conclusions about the problem under study and/or propose new focuses for such problems.

The goal would be to make the student capable of using the representation systems described in the analysis of variational problems. CAS or at least DGS with suitable graphic representations play a very important role for favoring observation and simulation to support to the comprehension of concepts and problem analysis.

The way as CAS or DGS are used in the classroom or outside the university will depend on the achievements to reach: (i) simple observation would correspond to a first level; (ii) a second stage implies simulation and observation; and (iii) one desirable level could be that from inferences, simulation and observation, that allow students to construct, by discovery, their own knowledge.

Additionally, to face the new curriculum of the bachelor in Applied Mathematics and Computing (UNAM, México) claims that CAS and DGS must be used in teaching process but the challenge is how can the faculty encourages the expecting changes?

Some of the math teachers have developed materials for exploring the possibilities of the use of CAS software and DGS in higher education.

Thus, we conceptualize our point of view through four possible uses in the classroom or outside the university in order to encourage the faculty to use them:

1. In the acquisition and / or consolidation of the concept of derivative through of sketching the graph of the derivative of a function with base only in the graph of the primitive function.
2. As a support for obtaining formulas, taking a particular case: the parametric equations of the cycloid.
3. As a support for determining the limits of integration in cylindrical and spherical coordinates that belongs to a 3D space.
4. As a support for understanding of the concept of singular solution and its enveloping.

## **1. Graphic analysis of the derivative of a function**

We have detected a learning problem with students of the first Calculus course. It consisted on finding out the function graph base on the derivative function graph data.

One of the mistakes showed by most of them was that they had assumed that the function graphic that had been showed in the problem would correspond to polynomials so the graphic of the derivative should have one concavity less than the original function.

Such a situation motivated us to make a Geogebra app that would assist students to understand the relationship between the derivative of a function graph and its primitive.

The app has two associated sliders. One of them permits the vertical displacement of the graphic over the ordinate axis. The other showed the values of the image when the function takes each value of its domain. In addition, the value of the slope of the tangent straight line was associated to each point that composed the function graphic. In like manner, the image was depicted when moving the slider (Figure 1. Curve colored in blue). Note the relation between the null slopes of the primitive function and the zeros of the derivative function. (The GeoGebra's algebraic view and the construction protocol are at annex 1).

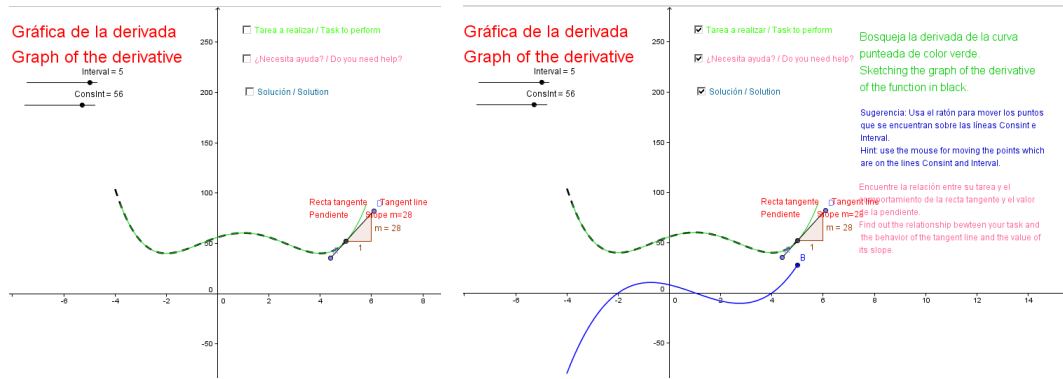


Figure 1. GeoGebra presentation as a learning object

## 2. Applications to integration limits on a region that belongs to a 3D space. Using CAS as helping for the task.

Among the difficulties students find in the study of real functions from Calculus of various variables,  $f: \mathbf{R} \rightarrow \mathbf{R}^3$ , it finds to discern the integration limits on a region that belongs to a three dimensional space. Even though, computer applications already exist to solve triple integrals, in these programs, the students should find out the suitable limits for solving these defined integrals, because the apps are not able to perform them.

Among the applications on triple integrals that share the same integration limits we can find the calculation of: volumes, mass moments and inertia moments. From this, the relevance of knowing how to pose in the correct way.

In the classroom to help students to recognize the integrations limits, we project on the blackboard 3D surfaces through graphics made with Maple or Maxima. On the same projection we sketch differential elements to get integration limits. For example, we propose to find out the integration limits in cylindrical coordinates for region between the sphere  $x^2 + y^2 + z^2 = 4^2$  and the cylinder  $x^2 + y^2 = 4x$  (Figures 2a & 2b) or the region inside the sphere  $x^2 + y^2 + z^2 = 4z$  and outside the cone  $x^2 + y^2 = z^2$  (Figure 2c). The

corresponding integrals are  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{4 \cos \theta} \int_0^4 f(r, \theta, z) r dz dr d\theta$  and  $\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{4 \cos \varphi} f(\rho, \varphi, \theta) \rho^2 \sin \varphi d\rho d\varphi d\theta$ .

(The GeoGebra's algebraic view and the construction protocol are at annex 2).

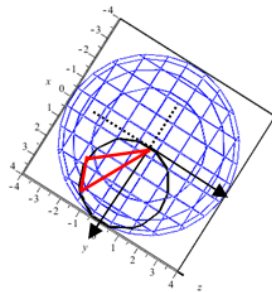
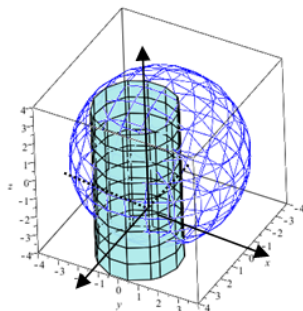


Figure 2a y 2b. Differential elements in cylindrical coordinates.

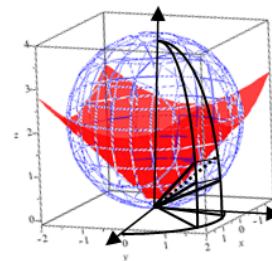


Figure 2c. Differential elements in spherical coordinates.

### 3. Applications to parametric functions. Using CAS as helping for finding out mathematical expressions

A cycloid is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line. This definition is almost the same from Mersenne approximately in 1615 [2]. Nevertheless, Galileo had already known the cycloid in 1599. The cycloid has been called "The Helen of Geometers" as it caused frequent quarrels among 17th-century mathematicians [3].

At the start of Vector Calculus course, we begin with geometry elements to show the advantages of using them in vector notation. So, we take up again parametric equations in functions as  $f: \mathbf{R} \rightarrow \mathbf{R}^n$ . A common example in textbooks corresponds to the cycloid, and the class task is to find the parametric equations for this one; see Figure 3. It means,  $x(\theta) = a(\theta - \sin \theta)$  and  $y(\theta) = a(1 - \cos \theta)$ . Though some textbooks give a graphic, this one is not the best, because the complementary angle is used.

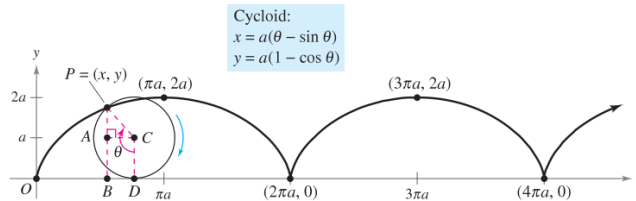


Figure 3. Description of the cycloid formation [4]

We propose to visualize the movement by a point on the rim of a circle when the circle rolls along a straight line and whose trace is the cycloid. (The GeoGebra's algebraic view and the construction protocol are at annex 3). This allows to get data that otherwise with the static graphic would be hard to figure out. First, to form the cycloid, a moving circle is showed; see Figure 4a. In a second stage, we add a triangle on the same circle; see Figure 4b.

Finally, some segments are added. These segments will permit to get the parametric expression of each variable, as function of parameter; see Figure 4c.

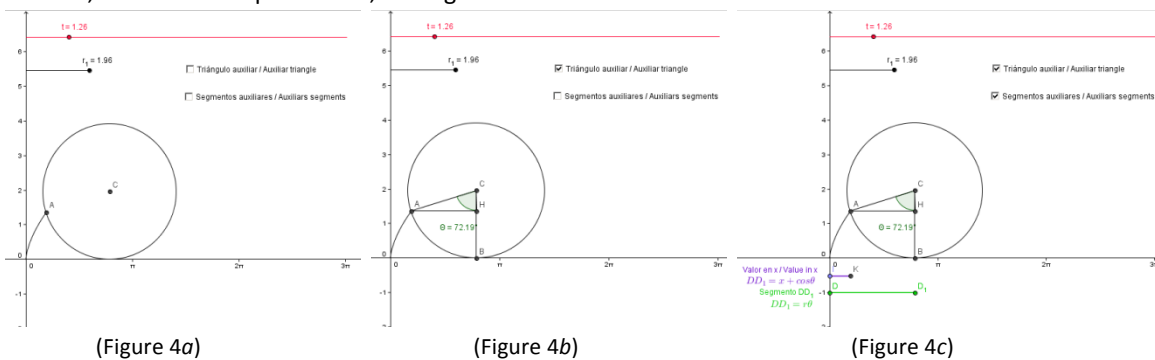


Figure 4. Sequence for obtaining the parametric equations of the cycloid graph.

### 4. Envelopes in Clairaut differential equation

The singular solution and general and particular solutions are concepts in a first course in differential equations.

We can find singular solution in equations of higher degree than one, as a Clairaut equation  $y = px + f(p)$  where  $p = \frac{dy}{dx}$ , and which its general solution is  $y = cx + f(c)$  where  $p = c$ , that is, the general solution is

a family of lines dependent on its parameter  $c$  (family of its integral curves). In order to find the singular solution [5], we use the Legendre transform to determine  $f = f(p)$  from  $y = y(x)$ . In other words, a relationship between the conventional point Pluecker<sup>4</sup> geometry refers to lines, with  $p \equiv \frac{dy}{dx}$  and

$$-x = \frac{df(p)}{dp}.$$

Suppose the differential equation of Clairaut  $\left(\frac{dy}{dx}\right)^2 + x \frac{dy}{dx} - y = 0$ . According the explained to previous lines, the general solution is:  $y = cx + c^2$  and the singular solution is:  $p^2 = y - px$  show in Figure 5.

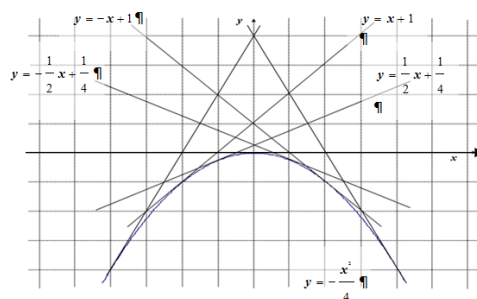


Figure 5. General solution (lines) and singular solution (parabole, the envelope)

The concept of singular solution enables us to capture geometric objects such as the envelope that has fallen into disuse due to curriculum design in some majors.

In fact, a learning problem has been detected in our students because it is hard to get the idea of a singular solution in touch in just a point of the general solution and at the same time being tangent to it.

This previous example pretend go beyond of context used as an easy motivation in basic education for doing handcrafts.

Another example is  $y - px - \frac{1}{p^2} = 0$ , and its general solution is:  $y = cx + \frac{1}{c^2}$ , whose singular solution is

$y = \frac{3x^{2/3}}{2^{2/3}}$ . The implementation of the app with Geogebra not only makes explicit the concept of the envelope but also lets see when the uniqueness of the solution fails at some initial value problem (Cauchy problem), v. gr.  $(x_0, y_0) = (0, 0)$ ; see Figure 6. Indeed, we can generalize that the failure of uniqueness to the initial value problem at each point on the curve; see Figure 7.

We must emphasize that a unique solution in each of its points, is tangent to a family member of a parameter that is the general solution; see Figure 8.

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<sup>4</sup> Pluecker geometry proposes a functional relationship for straight through the ordered pairs  $(p, f(p))$  where  $p$  is the slope and  $f(p)$  the intercept. For more information, see Callen, H. B. *Termodinámica*. Madrid: Editorial AC, 1981.

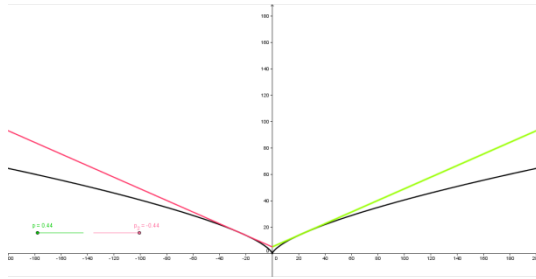


Figure 6. A failure of uniqueness to the initial value problem at the point  $(x_0, y_0) = (0, 0)$

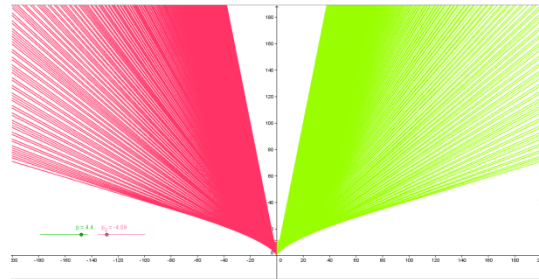


Figure 7. General solution

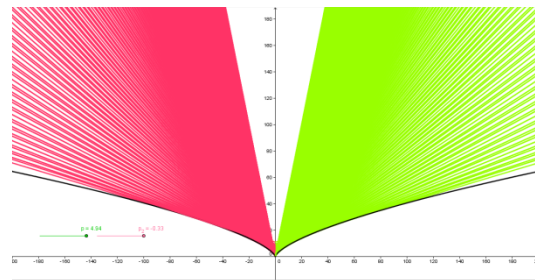


Figure 8. General solution and singular solution (envelope).  
Annex 4. Algebraic view, and construction protocol)

## Conclusions and reflections

- The computing activities developed in a personal way, allowed the students acquire enough competence to get the graphics of derivative functions, without resorting to any algorithmic of mathematical expressions. This seems to be an indicator of the development of variational thinking.
- The app for obtaining the graphic of the derivative function, we think it may be already consider as a learning object. The following stage will be to place the materials on a platform (i.e. Moodle) and to invite students to access the web and they can be capable of auto regulate their learning by themselves.
- The app about the cycloid has been taken to the classroom as a facilitator for the deduction of the cycloid parametric equations.
- By our experiences from the Faculty only the best students are able to produce their own knowledge without help from CAS. So, we have to give at others students opportunities for their own growing.
- The goal as a Math Professor: CAS learning not only for the students but also for professors.
- Our philosophy must be: "If you have already created or discovered something to improve the mathematical teaching then share it with your colleagues".

## References

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## Annex 1. Construction of the derivative functions with GeoGebra 4.2 as a learning object

### Algebra

### Construction protocol

Nº	Nombre	Definición	Valor	Subtítulo
1	Número Interval		Interval = 5	
2	Número Consint		Consint = 56	
3	Punto C	(Interval, Consint + 0.25Interval (Interval - 2) (Interval <sup>2</sup> - 2Interval - 16))	C = (5, 52.25)	
4	Curva i3	Curva[X, Consint + 0.25X <sup>4</sup> - X <sup>3</sup> - 3X <sup>2</sup> + 8X, X, -4, Interval]	i3: (X, 56 + 0.25X <sup>4</sup> - X <sup>3</sup> - 3X <sup>2</sup> + 8X)	
5	Curva c	Curva[X, 8 - 6X - 3X <sup>2</sup> + X <sup>3</sup> , X, -4, Interval]	c: (X, 8 - 6X - 3X <sup>2</sup> + X <sup>3</sup> )	
6	Punto B	(Interval, 8 - 6Interval - 3Interval <sup>2</sup> + Interval <sup>3</sup> )	B = (5, 28)	
7	Número m <sub>1</sub>	8 - 6Interval - 3Interval <sup>2</sup> + Interval <sup>3</sup>	m <sub>1</sub> = 28	
8	Texto texto3		"Gráfica de la derivada Graph of the derivative"	
9	Texto texto4		"Bosqueja la derivada de la curva punteada de color verde SK..."	
10	Texto texto2		"Sugerencia: Usa el ratón para mover los puntos que se enc..."	
11	Recta d	Tangente[C, i3]	d: y = 28x - 87.75	
12	Punto A	Punto sobre d	A = (4.4, 35.37)	
13	Texto texto1	"Recta tangente Tangent line Pendiente Slope m = m <sub>1</sub> + "	"Recta tangente Tangent line Pendiente Slope m = ..."	
14	Texto texto6		"Encuentre la relación entre su tarea y el comportamiento de l..."	
15	Función g	Función Consint + 0.25x <sup>4</sup> - x <sup>3</sup> - 3x <sup>2</sup> + 8x en el intervalo [-3, 8, 5, 8]	g(x) = 56 + 0.25x <sup>4</sup> - x <sup>3</sup> - 3x <sup>2</sup> + 8x	
16	Valor Booleano a		a = false	Tarea a realizar / Task to perform
17	Valor Booleano b		b = false	¿Necesita ayuda? / Do you need help?
18	Valor Booleano f		f = false	Solución / Solution
19	Número m	Pendiente de d	m = 28	
20	Punto D	Punto sobre d	D = (6.07, 82.35)	
21	Segmento e	Segmento [A, D]	e = 47.01	



Annex 2. Construction of 3D graphs with MAPLE 15.

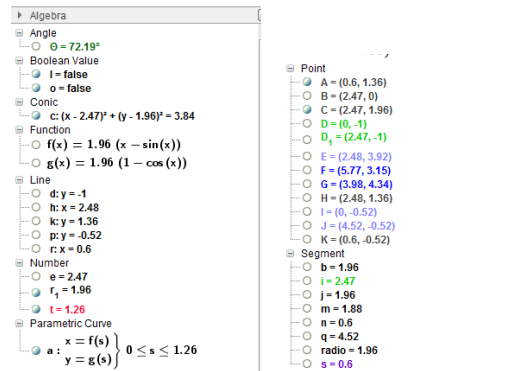
with(plots) :

```
implicitplot3d([x^2 + y^2 + z^2 = 4·z, x^2 + y^2 = z^2], x=-2..2, y=-2..2, z
=-0.1..4, style = [line, surface], color = [blue, red], scaling
= constrained, transparency = [0.0, 0.25], axes = boxed);
```

```
implicitplot3d([x^2 + y^2 + z^2 = 16, 4·x = x^2 + y^2], x=-4..4, y=-4..4, z
=-4..4, style = [line, surface], color = [blue, turquoise], scaling
= constrained, transparency = [0.0, 0.25], axes = boxed);
```

Annex 3. Construction of the cycloid with GeoGebra 4.2 for deduction of parametric equations

Algebra



Construction protocol

No.	Name	Definition	Value	Caption
1	Number r <sub>1</sub>		r <sub>1</sub> = 1.96	
2	Number t		t = 1.26	
3	Function f	f(x) = r <sub>1</sub> · (x - sin(x))	f(x) = 1.96 (x - ...	
4	Function g	g(x) = r <sub>1</sub> · (1 - cos(x))	g(x) = 1.96 (1 ...	
5	Curve a	Curve f(s), g(s), s, 0, t	a(f(s), g(s))	
6	Point A	a(t)	A = (0.6, 1.36)	
7	Point B	(r <sub>1</sub> , t, 0)	B = (2.47, 0)	
8	Point C	(r <sub>1</sub> , t, r <sub>1</sub> )	C = (2.47, 1.96)	
9	Segment...	Segment [C, A]	radio = 1.96	
10	Circle c	Circle through B with center C	c: (x - 2.47)^2 + ...	
11	Segment...	Segment [C, B]	b = 1.96	
12	Point D	Point on yAxis	D = (0, -1)	
13	Line d	Line through D perpendicular	d: y = -1	
14	Number e	Distance of (0, 0) and B	e = 2.47	
15	Point D <sub>1</sub>	(e, -1)	D <sub>1</sub> = (2.47, -1)	
16	Segment i	Segment [D, D <sub>1</sub> ]	i = 2.47	
17	Text texto1		"Segmento D...	
18	Angle theta	Angle between A, C, B	theta = 72.19°	
19	Point F		F = (5.77, 3.15)	
20	Point G		G = (3.98, 4.34)	
21	Point E	Point on c	E = (2.48, 3.92)	
22	Line h	Line through E perpendicular	h: x = 2.48	
23	Segment j	Segment [C, E]	j = 1.96	
24	Line k	Line through A parallel to i	k: y = 1.36	
25	Point H	Intersection point of h, k	H = (2.48, 1.36)	
26	Segment m	Segment [A, H]	m = 1.88	
27	Segment n	Segment [H, C]	n = 0.6	
28	Point l	Point on yAxis	l = (0, -0.52)	
29	Line p	Line through l perpendicular	p: y = -0.52	
30	Point J	Point on p	J = (4.52, -0.52)	
31	Segment q	Segment [J, l]	q = 4.52	
32	Line r	Line through A perpendicular	r: x = 0.6	
33	Point K	Intersection point of r, q	K = (0.6, -0.52)	
34	Segment s	Segment [l, K]	s = 0.6	
35	Boolean i		i = false	Triángulo auxiliar i
36	Text texto1		"Valor en x / V...	
37	Boolean o		o = false	Segmentos auxiliares f
38	Text texto2		"\$ DD_1=r\$the...	
39	Text texto3		"\$ DD_1=x+c...	

Annex 4. Construction of the envelope curve with GeoGebra 4.2 for a Clairaut Differential Equation.

Algebra

Function

- $f(x) = 0.44x + \frac{1}{0.44^2}$
- $g(x) = \frac{3x^3}{2^3}$
- $h(x) = -0.44x + \frac{1}{(-0.44)^2}$

Number

- $p = 0.44$
- $p_2 = -0.44$

Construction protocol

No.	Name	Definition	Value	Caption
1	Number p		$p = 0.44$	
2	Function g	Function $(3x^3 / 3) / 2^3$	$g(x) = (3x^3 / 3) / 2^3$	
3	Number $p_2$		$p_2 = -0.44$	
4	Function f	Function $p x + 1 / p^2$ on interval	$f(x) = 0.44x + 1 / 0.44^2$	
5	Function h	Function $p_2 x + 1 / p_2^2$ on interval	$h(x) = -0.44x + 1 / (-0.44)^2$	