

Motivating students in an introductory matrix algebra course



In recreational mathematics, a semi-magic (magic) square of order n is an arrangement of n^2 numbers, usually distinct integers, in a square, such that the n numbers in all rows & all columns (& both diagonals) add up to the same constant s , called its magic sum.

A normal magic square of order n contains the integers from 1 to n^2 .

History I



Luoshu,
China,
650 BC

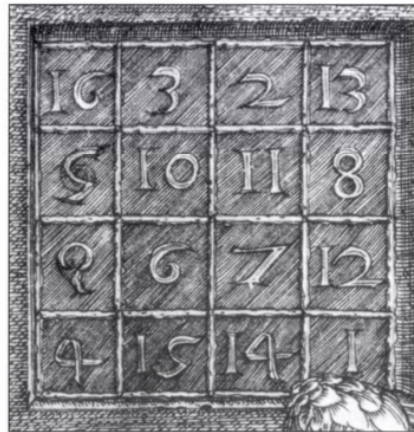
$$L = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$$

History II



Melencolia I
by
Albrecht
Dürer,
1514

History II



$$D = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

History III



Basilica de
la Sagrada
Familia,
Barcelona,
Spain
(1882-2026)

History III



$$S = \begin{pmatrix} 1 & 14 & 14 & 4 \\ 11 & 7 & 6 & 9 \\ 8 & 10 & 10 & 5 \\ 13 & 2 & 3 & 15 \end{pmatrix}$$

Fachada de la Pasión (by Josep Subirachs):
El beso de Judas

Is L really magic? (matrix operations and trace I)

$$L = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}; \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad F = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$L\mathbf{1} = \begin{pmatrix} 15 \\ 15 \\ 15 \end{pmatrix} = s\mathbf{1} \quad \mathbf{1}'L = (15 \quad 15 \quad 15) = s\mathbf{1}'$$

$$\text{tr}(L) = 15 \quad \text{tr}(FL) = \begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix} = 15$$

Is D really magic? (matrix operations and trace II)

$$D = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}; \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad F = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$D\mathbf{1} = \begin{pmatrix} 34 \\ 34 \\ 34 \\ 34 \end{pmatrix}; \quad \mathbf{1}'D = (34 \quad 34 \quad 34 \quad 34)$$

$$\text{tr}(D) = 34; \quad \text{tr}(FD) = \begin{pmatrix} 4 & 15 & 14 & 1 \\ 9 & 6 & 7 & 12 \\ 5 & 10 & 11 & 8 \\ 16 & 3 & 2 & 13 \end{pmatrix} = 34$$

Is S really magic? (matrix operations and trace III)

$$S = \begin{pmatrix} 1 & 14 & 14 & 4 \\ 11 & 7 & 6 & 9 \\ 8 & 10 & 10 & 5 \\ 13 & 2 & 3 & 15 \end{pmatrix}; \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad F = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$S\mathbf{1} = \begin{pmatrix} 33 \\ 33 \\ 33 \\ 33 \end{pmatrix}; \quad \mathbf{1}'S = (33 \quad 33 \quad 33 \quad 33)$$

$$\text{tr}(S) = 33; \quad \text{tr}(FS) = 33$$

$Ax = \lambda x$ (eigenvalues and -vectors)

M , $n \times n$ matrix denoting magic square

$\mathbf{1}$, $n \times 1$ vector of ones

s , magic sum of M

$$M\mathbf{1} = s\mathbf{1}$$

eigenvector

eigenvalue

Examples: L , D , S

$$\text{EIGENVALUES}(L) = [15, \sqrt{24} \cdot i, -\sqrt{24} \cdot i]$$

$$\text{EIGENVALUES}(D) = [34, 0, 8, -8]$$

$$\text{EIGENVALUES}(S) = [33, \approx 0.8445, \approx -7.734, \approx 6.890]$$

Parameterization of 3x3 magic squares I

$$\underset{3 \times 3}{M} = sG + N; \quad G = \frac{1}{3}\mathbf{1}\mathbf{1}' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}; \quad N = \begin{pmatrix} \alpha + \beta & -2\alpha & \alpha - \beta \\ -2\beta & 0 & 2\beta \\ -\alpha + \beta & 2\alpha & -\alpha - \beta \end{pmatrix}$$

Example: L

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#5: M := s * G + N
#6: L := [ [ 4 9 2 ]
           [ 3 5 7 ]
           [ 8 1 6 ] ]
#7: M = L
#8: SOLVE(M = L, [s, alpha, beta])
#9: s = 15 ^ alpha = -2 ^ beta = 1
  
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Parameterization of 3x3 magic squares II

$$\mathbf{M}_{3 \times 3} = s\mathbf{G} + \mathbf{N}; \quad \mathbf{G} = \frac{1}{3}\mathbf{1}\mathbf{1}' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}; \quad \mathbf{N} = \begin{pmatrix} \alpha + \beta & -2\alpha & \alpha - \beta \\ -2\beta & 0 & 2\beta \\ -\alpha + \beta & 2\alpha & -\alpha - \beta \end{pmatrix}$$

	trace	det	rank	eigenvalues
G	1	0	1	[1, 0, 0]
N	0	0	2	[0, $\sqrt{12\alpha\beta}$, $-\sqrt{12\alpha\beta}$]
M	s	- s · 12αβ	3	[s, $\sqrt{12\alpha\beta}$, $-\sqrt{12\alpha\beta}$]
L	15	360	3	[15, $\sqrt{24} \cdot i$, $-\sqrt{24} \cdot i$]

The 3 examples again (trace, det, rank, eigenvalues)

	trace	det	rank	eigenvalues
L	15	360	3	$[15, \sqrt{24}\cdot i, -\sqrt{24}\cdot i]$
D	34	0	3	$[34, 0, 8, -8]$
S	33	-1485	4	$[33, \approx 0.8445, \approx -7.734, \approx 6.890]$

Inverses I

L is a nonsingular matrix

$$L^{-1} = \begin{pmatrix} \frac{23}{360} & -\frac{13}{90} & \frac{53}{360} \\ \frac{19}{180} & \frac{1}{45} & -\frac{11}{180} \\ -\frac{37}{360} & \frac{17}{90} & -\frac{7}{360} \end{pmatrix}$$

$$L^{-1}\mathbf{1} = \begin{pmatrix} \frac{1}{15} \\ \frac{1}{15} \\ \frac{1}{15} \end{pmatrix}$$

$$\mathbf{1}'L^{-1} = \left(\frac{1}{15} \quad \frac{1}{15} \quad \frac{1}{15} \right)$$

	trace	det	rank	eigenvalues
L	15	360	3	$[15, \sqrt{24}\cdot i, -\sqrt{24}\cdot i]$
L^{-1}	1/15	1/360	3	$[1/15, 1/(-\sqrt{24}\cdot i), 1/(\sqrt{24}\cdot i)]$

$$\text{tr}(FL^{-1}) = \frac{1}{15}$$

L^{-1} is magic.

Inverses II

D is a singular matrix

$$D^+ = \begin{pmatrix} \frac{55}{544} & -\frac{201}{2720} & -\frac{167}{2720} & \frac{173}{2720} \\ \frac{37}{2720} & -\frac{31}{2720} & -\frac{13}{544} & \frac{139}{2720} \\ -\frac{99}{2720} & \frac{21}{544} & \frac{71}{2720} & \frac{3}{2720} \\ -\frac{133}{2720} & \frac{207}{2720} & \frac{241}{2720} & -\frac{47}{544} \end{pmatrix} \quad D^+ \mathbf{1} = \begin{pmatrix} \frac{1}{34} \\ \frac{1}{34} \\ \frac{1}{34} \\ \frac{1}{34} \end{pmatrix}$$

$$\mathbf{1}' D^+ = \left(\frac{1}{34} \quad \frac{1}{34} \quad \frac{1}{34} \quad \frac{1}{34} \right)$$

(D^+ denotes the Moore-Penrose inverse of D)

	trace	det	rank	eigenvalues
D	34	0	3	[34, 0, 8, -8]
D^+	1/34	0	3	[1/34, 0, 1/10, -1/10]

$$\text{tr}(FD^+) = \frac{1}{34}$$

D^+ is magic.

Inverses III

S is a nonsingular matrix

$$S^{-1} = \begin{pmatrix} -\frac{13}{99} & -\frac{2}{99} & \frac{20}{99} & -\frac{2}{99} \\ \frac{67}{495} & \frac{353}{495} & -\frac{263}{495} & -\frac{142}{495} \\ -\frac{43}{495} & -\frac{362}{495} & \frac{287}{495} & \frac{133}{495} \\ \frac{56}{495} & \frac{34}{495} & -\frac{109}{495} & \frac{34}{495} \end{pmatrix} \quad S^{-1}\mathbf{1} = \begin{pmatrix} \frac{1}{33} \\ \frac{1}{33} \\ \frac{1}{33} \\ \frac{1}{33} \end{pmatrix}$$

$$\mathbf{1}'S^{-1} = \left(\frac{1}{33} \quad \frac{1}{33} \quad \frac{1}{33} \quad \frac{1}{33} \right)$$

	trace	det	rank	eigenvalues
S	33	-1485	4	[33, ≈0.8445, ≈-7.734, ≈6.890]
S⁻¹	203/ 165	-1/1485	4	[1/33, ≈1/0.8445, ≈-1/7.734, ≈1/6.890]

$$\text{tr}(FS^{-1}) = -\frac{193}{165}$$

S⁻¹ is only semi-magic!

References

- K. Schmidt and G. Trenkler, *The Moore-Penrose Inverse of a Semi-Magic Square is Semi-Magic*, **International Journal of Mathematical Education in Science and Technology** 32 (2001), 624-629.
- G. Trenkler, K. Schmidt, and D. Trenkler, *A Simple Parameterization of 3x3 Magic Squares*, **International Journal of Mathematical Education in Science and Technology** 43 (2012), 128-133.