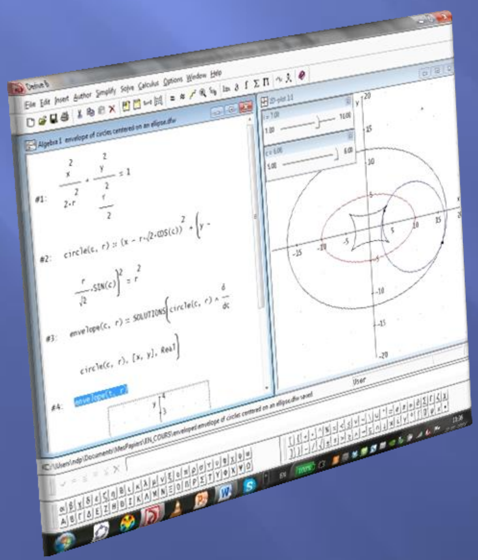
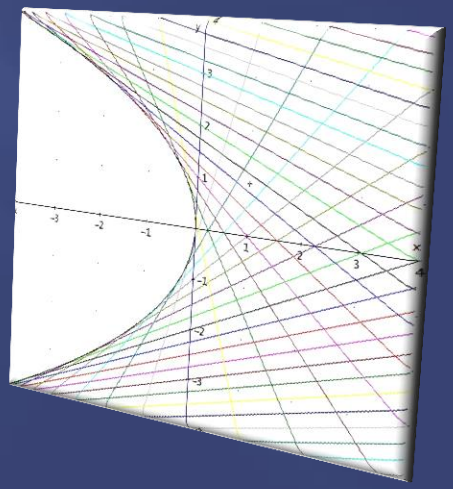
THE STUDY OF ENVELOPES IN A CAS ENVIRONMENT

Th. Dana-Picard – G. Mann – N. Zehavi

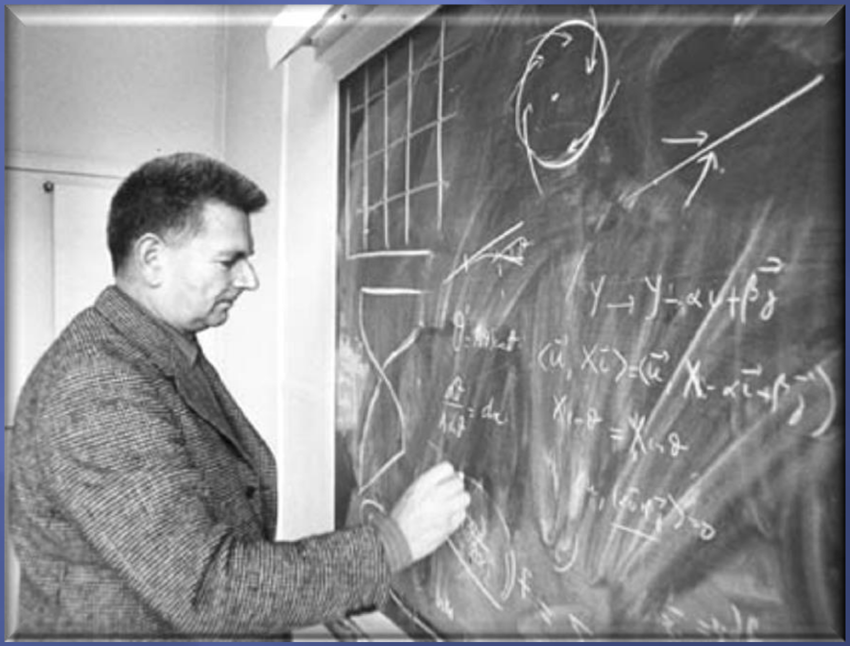


July 3rd, 2014



Motivation

René Thom: Sur la théorie des enveloppes. J. Math. Pures et Appliquées, XLI,fasc. 2, 1962, pp. 177-192, « manuscrit reçu le 25 avril 1960 ».



R. Thom

La récente réforme des études de licence en mathématiques a complètement évincé des programmes la théorie des enveloppes. … je ne puis que trouver cette disparition très regrettable; rappelons … le rôle des enveloppes dans la théorie des équations différentielles (intégrales singulières), et des équations aux dérivées partielles; mais est-il concevable qu'un professeur de lycée ait quelque usage des problèmes de la Géométrie Elementaire, sans connaître …. les phénomènes généraux de cette théorie?

R. Thom

Même d'un point de vue pratique, la théorie des enveloppes rend compte de phénomènes familiers, sans elle inexpliqués; pour s'en convaincre, il suffit d'observer, à l'intérieur d'un bol hémisphérique de café au lait convenablement éclairé, la structure cuspidale des caustiques de réflexion, et leur variation quand l'éclairage se modifie.

The reason of the disappearance, according to Thom

1. The classical theory is not rigorous enough
2. Too many particular cases: fixed points, singular points, stationary curves, etc.
3. Nothing ensures that all the “pathological” cases have been included in a catalogue
4. Actually: the theory is so rich that it is impossible to force it into the rules of a rigorous pedagogy.

Introduction

1. A short review of History
2. First examples
3. The MA program for in-service teachers

Curvature

Consider a curve given by a parameterization by arc length s, i.e. $\mathbf{r}=\mathbf{r}(s)=[x(s),y(s)]$. The parameterization defines a direction on the curve, as s increases.

Define

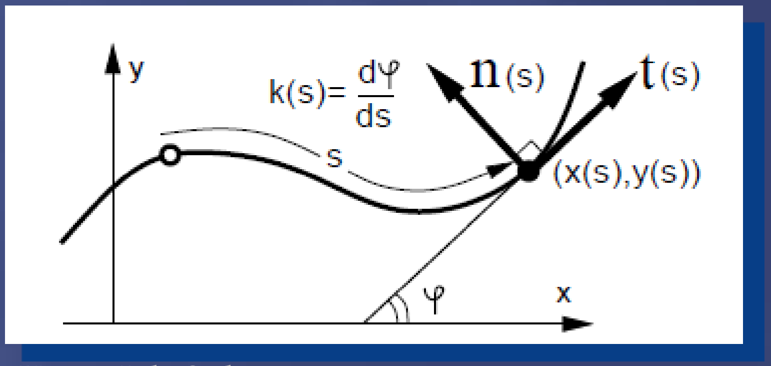
$\mathbf{t}(s)$ = the unit tangent vector associated with the direction

n(s) = the unit normal vector s.t. (**t,n**) is a counterclockwise oriented frame.

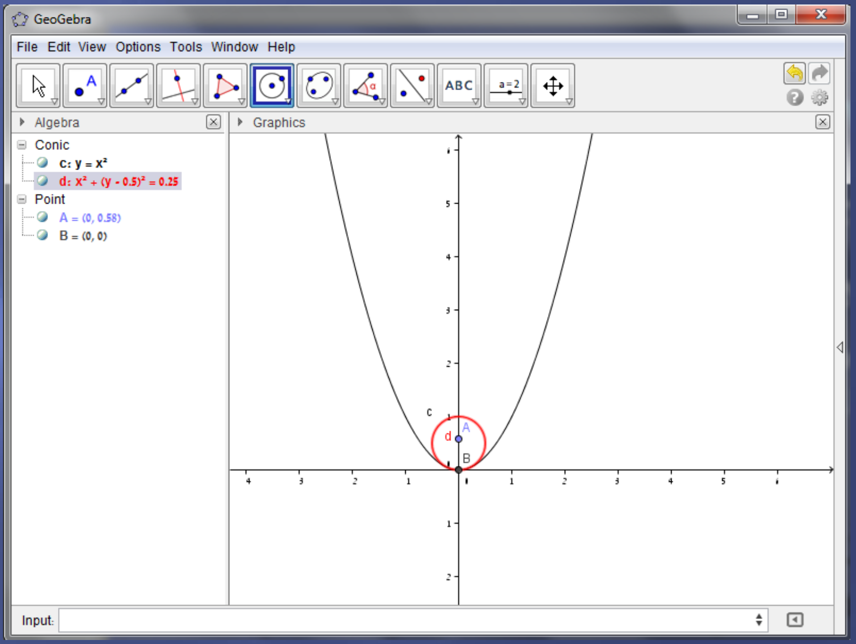
$\varphi(s)$ = the angle of the tangent with the positive x-axis

The **curvature** at a point measures the **rate of curving as the point moves along the curve with unit speed**

$$k(s) = \frac{d\varphi(s)}{ds}$$



- The curvature of a straight line is 0 everywhere
- On a curve, points where $k=0$ are points of inflexion.
- At every point on a circle, the curvature of the circle oriented by its inner normal is the reciprocal of the radius of the circle .
- The radius of curvature of a curve at a point $[x(s),y(s)]$ is equal to $1/k(s)$.



Curvature of a plane curve

PARAMETRIC CURVE

- The curve is given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}$
- Then:

$$\kappa = \frac{\left|\dot{x}\ddot{y}-\dot{y}\ddot{x}\right|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}$$

GRAPH OF A FUNCTION

The curve is given by $y=f(x)$.
The corresponding vector formula is r(x)=x **i** + $f(x)$ **j**.
Then:

$$\kappa(x) = \frac{\left|f''(x)\right|}{\left[1+\left(f'(x)\right)^2\right]^{3/2}}$$

Reconstruction of a curve from its curvature

We have: $$\frac{d\mathrm{r}(s)}{ds} = \mathrm{t}(s) = [\cos\varphi(s), \sin\varphi(s)]$$

$$k(s) = \frac{d\varphi(s)}{ds}$$

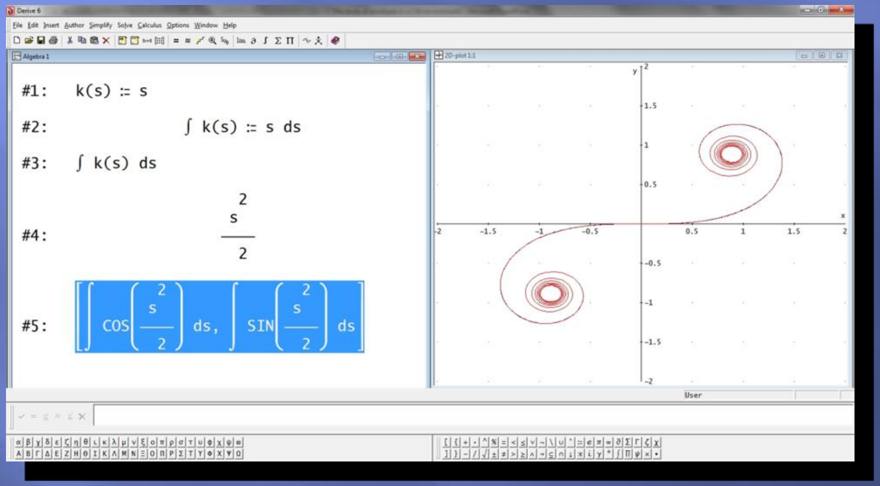
We can reconstruct the curve from its curvature, up to rigid transformations (determined by 3 constants):

$$\varphi(s) = \int k(s)ds + \varphi_0$$

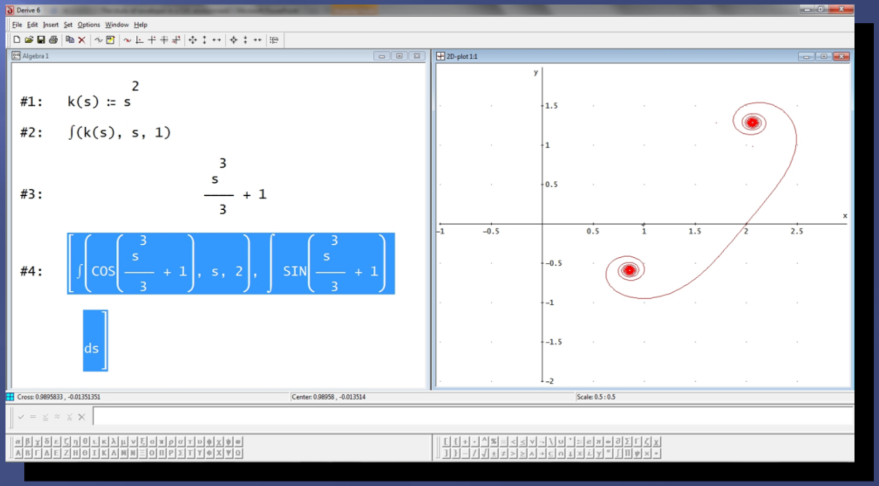
$$\mathrm{r}(s) = \left[\int \cos\varphi(s)\,ds + a, \int \sin\varphi(s)\,ds + b\right]$$

Examples of reconstruction

CONSTANTS=0



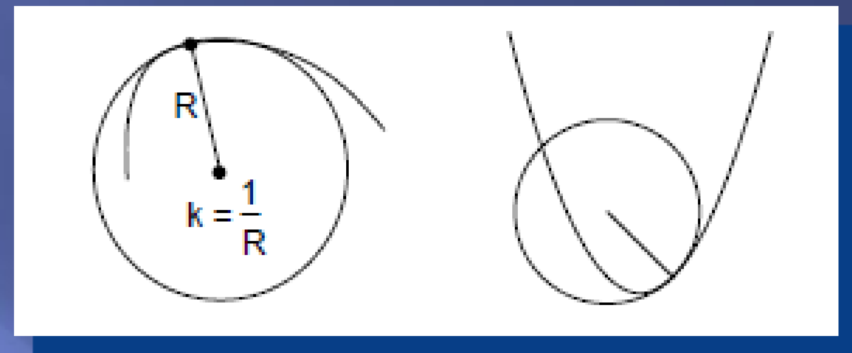
NON ZERO CONSTANTS

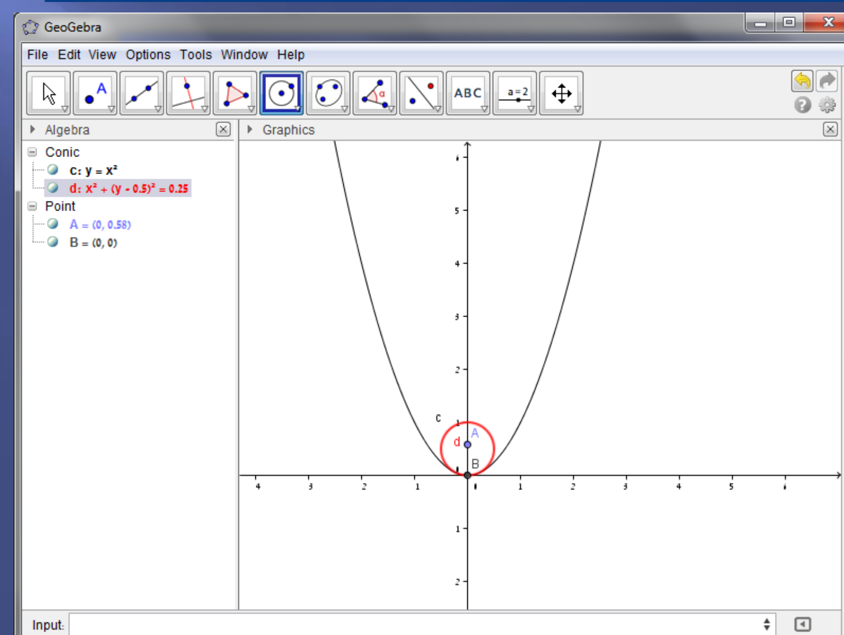


Circle of curvature

The **circle of curvature** at non-inflexion point P (i.e. $k \neq 0$) on a plane curve is the circle such that:

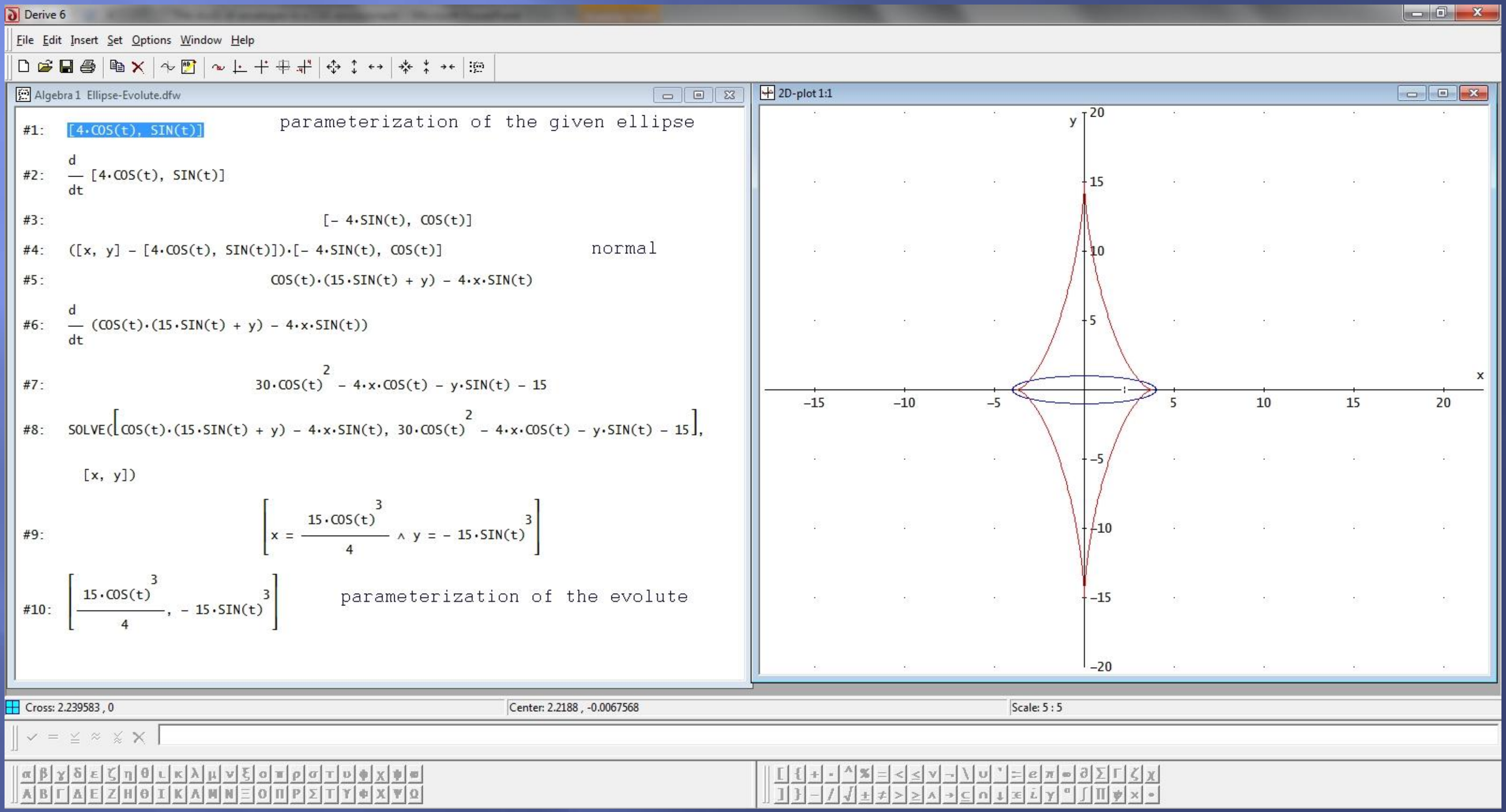
- It is tangent to the curve at P
- It has the same curvature as the curve at P
- It lies toward the concave (or inner) side of the curve
- The center of the circle of curvature at P is called the **center of curvature** at P.





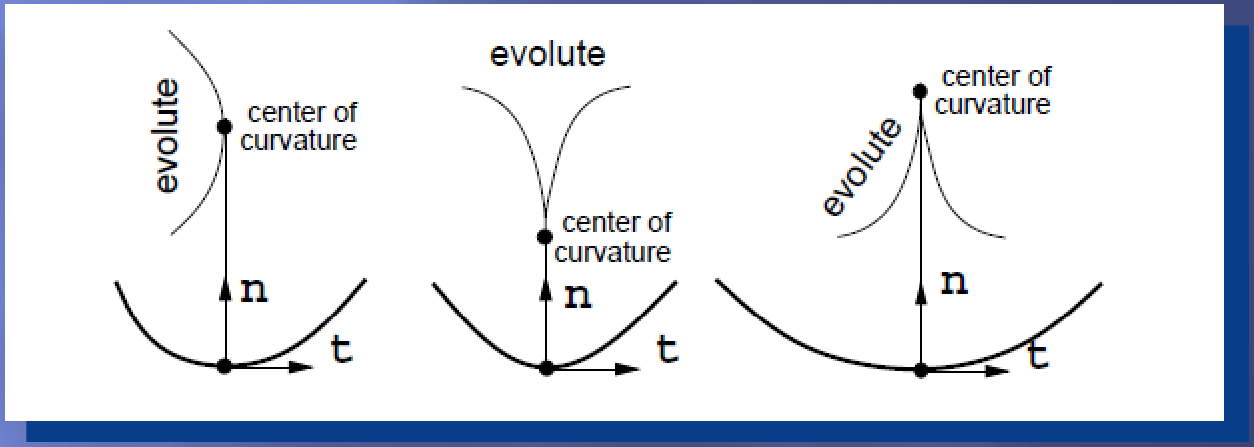
Evolute

The geometric locus of the centers of curvature of a given curve C is called the **evolute** of C.



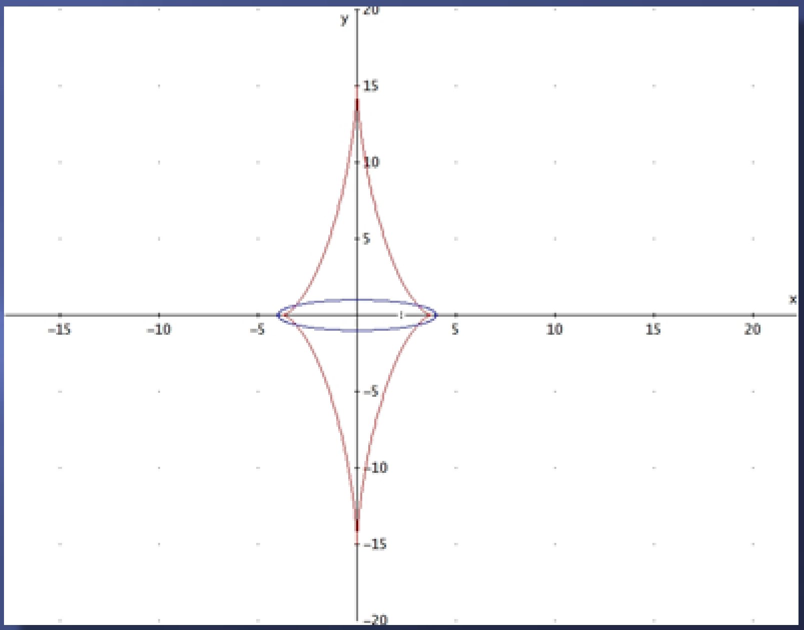
From normals to Evolute

<u>Thm</u>: The evolute is tangent to the normals to the given curve at the centers of curvature



Cusps on the evolute

<u>Thm:</u> The evolute of a curve C: $r=r(t)$ has a cusp at the point $r(t_0)$ if, and only if, $r(t_0)$ is a vertex of C. The cusp on the evolute is pointing towards or away from the vertex according to the absolute value of the curvature at $r(t_0)$ having an absolute minimum or maximum at $r(t_0)$.



Algebra 1 ParabolaEvolutes.dfw

#1: $\left[\cos(t),\ \cos(t)^2\right]$

#2: $\frac{d}{dt}\left[\cos(t),\ \cos(t)^2\right]$

#3: $[-\sin(t),\ -2\cdot\sin(t)\cdot\cos(t)]$

equation of normals to the curve

#4: $([x,\ y]-\left[\cos(t),\ \cos(t)^2\right])\cdot[-\sin(t),\ -2\cdot\sin(t)\cdot\cos(t)]$

#5: $2\cdot\sin(t)\cdot\cos(t)^3+(1-2\cdot y)\cdot\sin(t)\cdot\cos(t)-x\cdot\sin(t)$

reduced equation of the normals

#6: $y=\cos(t)^2-\frac{x}{2\cdot\cos(t)}+\frac{1}{2}$

#7: $\mathrm{VECTOR}\left(\cos(t)^2-\frac{x}{2\cdot\cos(t)}+\frac{1}{2},\ t,\ 0,\ 3.14,\ 0.1\right)$

reduced form for the equation of the normals

#8: $\mathrm{SOLVE}(2\cdot\sin(t)\cdot\cos(t)^3+(1-2\cdot y)\cdot\sin(t)\cdot\cos(t)-x\cdot\sin(t),\ y)$

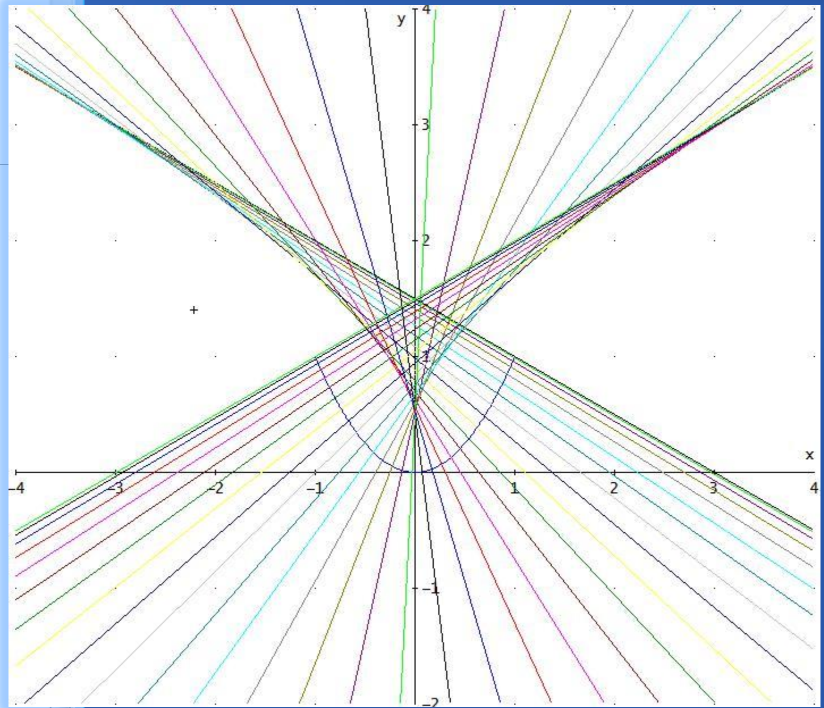
#9: $y=\cos(t)^2-\frac{x}{2\cdot\cos(t)}+\frac{1}{2}$

#10: $\frac{d}{dt}(2\cdot\sin(t)\cdot\cos(t)^3+(1-2\cdot y)\cdot\sin(t)\cdot\cos(t)-x\cdot\sin(t))$

parameterization of the evolute

#11: $\mathrm{SOLVE}(\left[2\cdot\sin(t)\cdot\cos(t)^3+(1-2\cdot y)\cdot\sin(t)\cdot\cos(t)-x\cdot\sin(t),\ 8\cdot\cos(t)^4-4\cdot(y+1)\cdot\cos(t)^2-x\cdot\cos(t)+2\cdot y-1\right],\ [x,\ y])$

#12: $\left[\frac{6\cdot\cos(t)^5}{\sin(t)^2}+\cos(t)^3\cdot\left(2-\frac{5}{\sin(t)^2}\right)+\cos(t)\cdot\left(1-\frac{1}{\sin(t)^2}\right),\ \frac{3\cdot\cos(t)^4}{\sin(t)^2}+\frac{5\cdot\cot(t)^2}{2}+\frac{1}{2\cdot\sin(t)^2}\right]$



Envelopes

General definition of the envelope of a family of plane curves

Consider a parameterized family *F* of plane curves , dependent on a real parameter *k*. We denote by an equation for the curves in the family *F*. A plane curve *E* is called **an envelope** of the family *F* if the following properties hold:

(i) every curve is tangent to *E;*

(ii) to every point *M* on *E* is associated a value *k*(*M*) of the parameter *k,* such that is tangent to *E* at the point *M;*

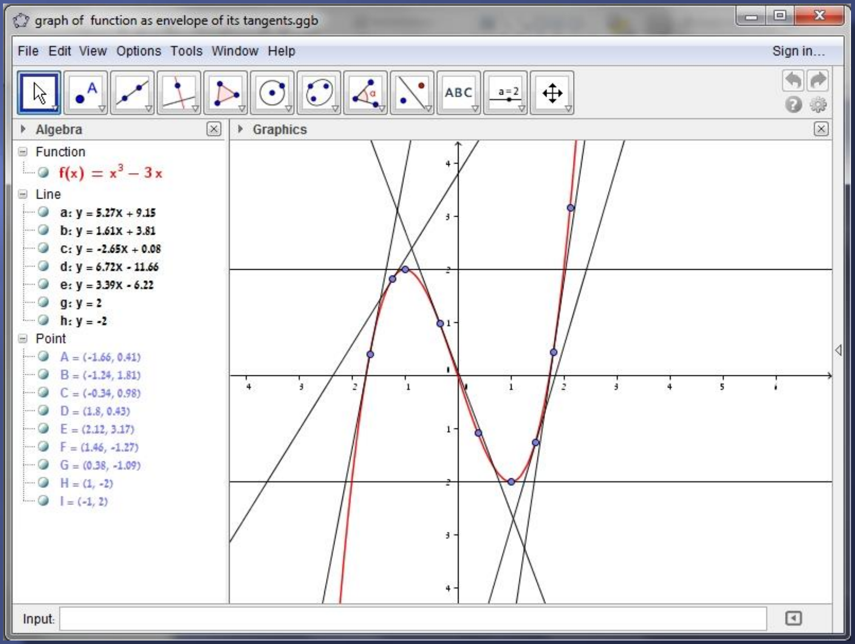
(iii) The function *k*(*M*) is non-constant on every arc of *E*.

First examples

1. **The tangents to a given curve**
2. The family of lines defined by 1,c,c^2
3. The dynamical example of a (family of) parabola(s) sliding on a parabola

Tangents to a given curve

The graph of a smooth function can be viewed as the envelope of its tangents



First examples

1. The tangents to a given curve
2. **The family of lines defined by 1,c,c^2**
3. The dynamical example of a (family of) parabola(s) sliding on a parabola

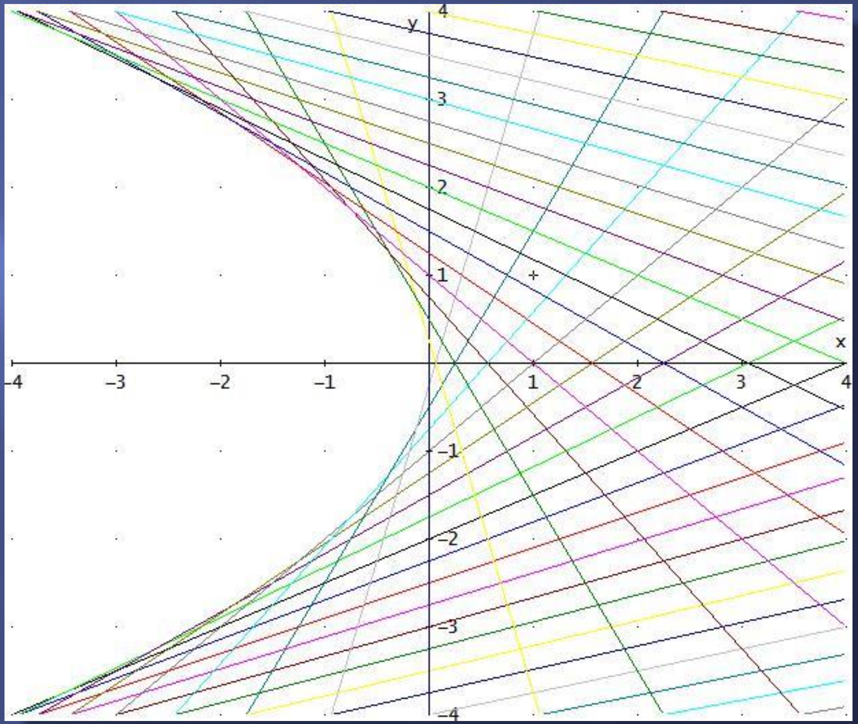
A geometric progression family of lines : algebraic treatment

We consider the 1-parameter family F of lines given by the equations

$$x + cy = c^2$$

where c is a real parameter.

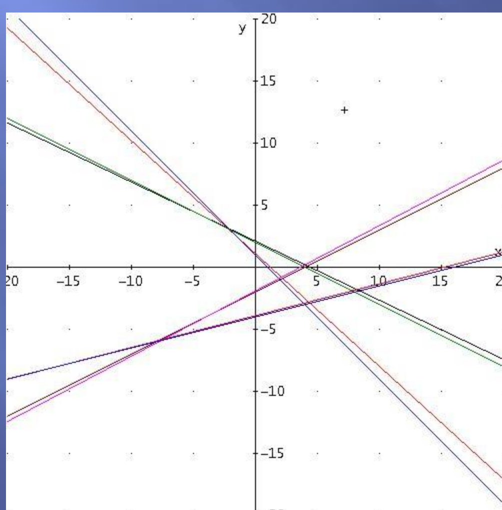
1. We **conjecture** **that an envelope is the parabola whose equation is** $x = -\frac{1}{4}y^2$
2. **We check this graphically.**
3. **We check this algebraically.**

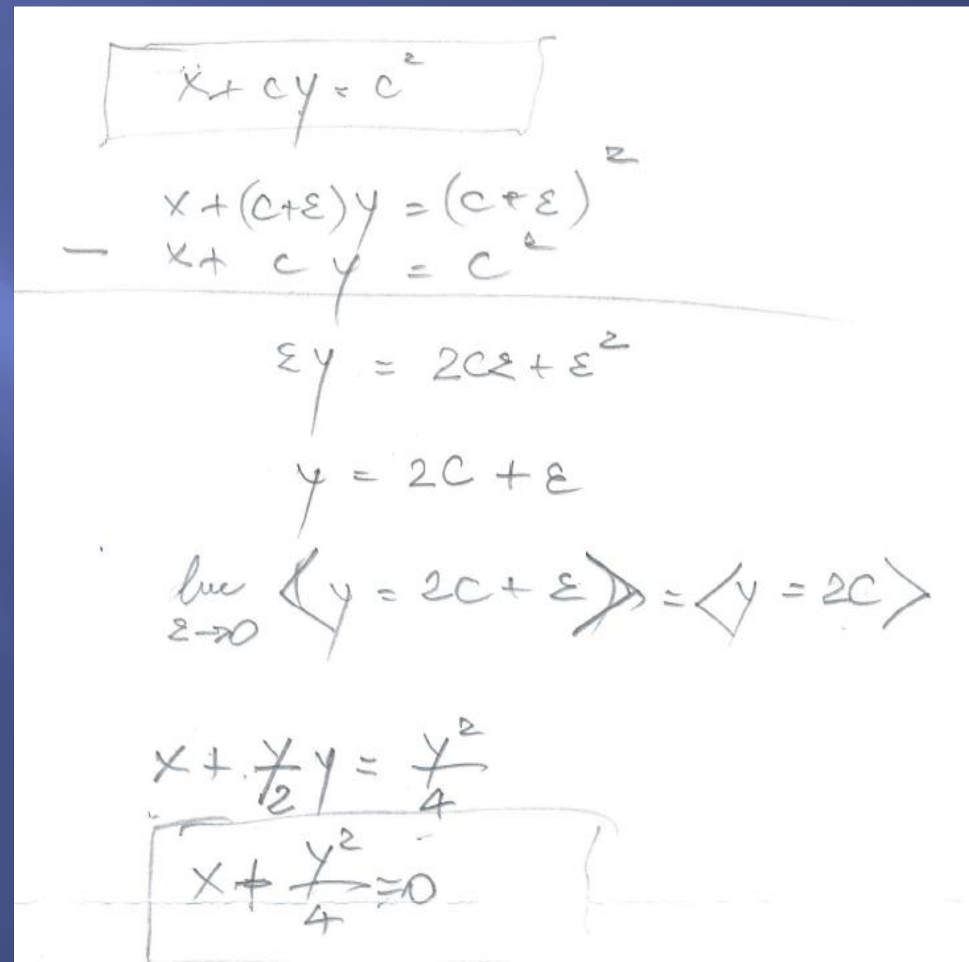


Same example: infinitesimally close lines

We consider the 1-parameter family F of lines given by the equations $x + cy = c^2$

where *c* is a real parameter.





Solving the system of equations [f(x,y,c)=0 and der(f(x,y,c),c)=0]

An envelope of the family is (a *subset* of) the curve defined by the following equations:

$$\begin{cases} f(x,y,c)=0 \\ \dfrac{\partial}{\partial c} f(x,y,c)=0 \end{cases}$$

Proof:

$$(*)\ f(x,y,c)=0$$

$$f(x,y,c+\varepsilon)=0$$

$$f(x,y,c+\varepsilon)-f(x,y,c)=0$$

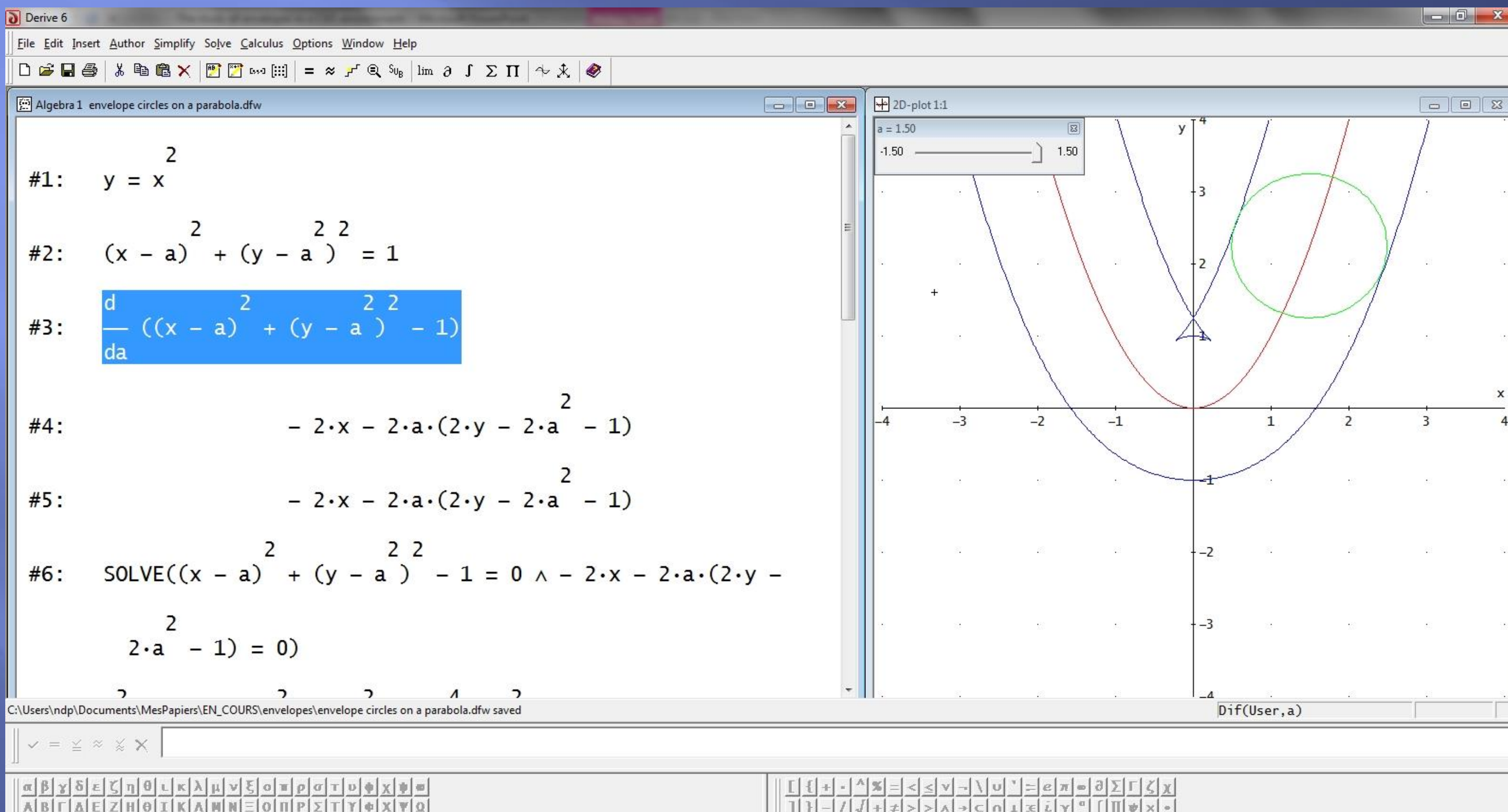
$$\frac{f(x,y,(c+\varepsilon)-f(x,y,c)}{\varepsilon}=0$$

$$\lim \langle \quad \rangle = \left\langle \frac{\partial f(x,y,c)}{\partial c}=0 \right\rangle (**)$$

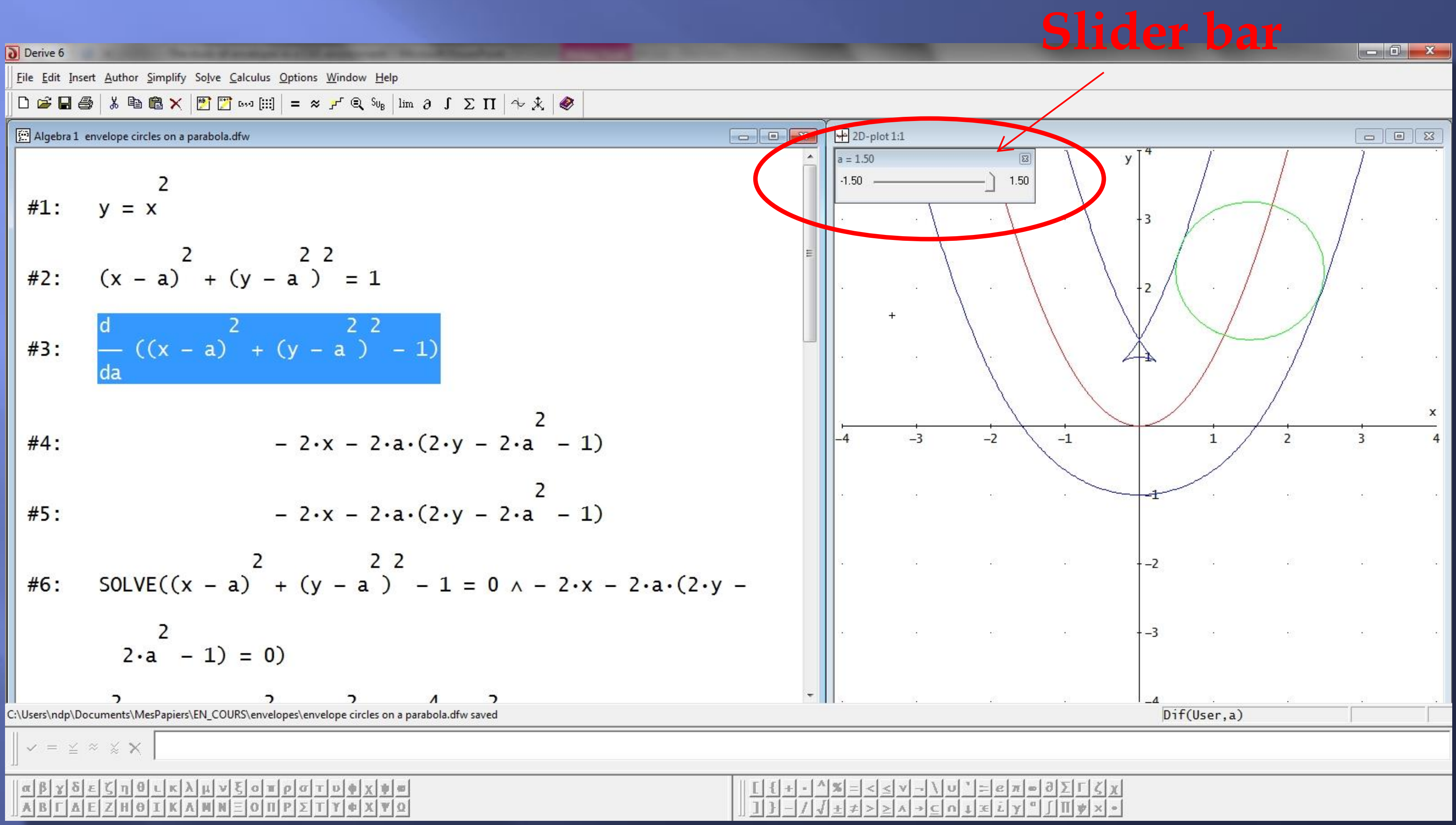
First examples

1. The tangents to a given curve
2. The family of lines defined by 1,c,c^2
3. **The dynamical example of a (family of) parabola(s) sliding on a parabola**

A family of circles sliding on a parabola



A family of circles sliding on a parabola



Circles whose centers are on a given ellipse

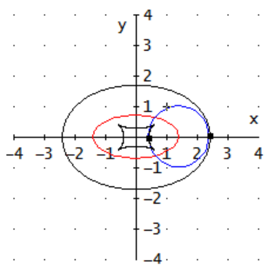
Algebra 1 envelope of circles centered on an ellipse - one parameter.dfw

#1: $$\frac{x^2}{2} + \frac{y^2}{\frac{1}{2}} = 1$$

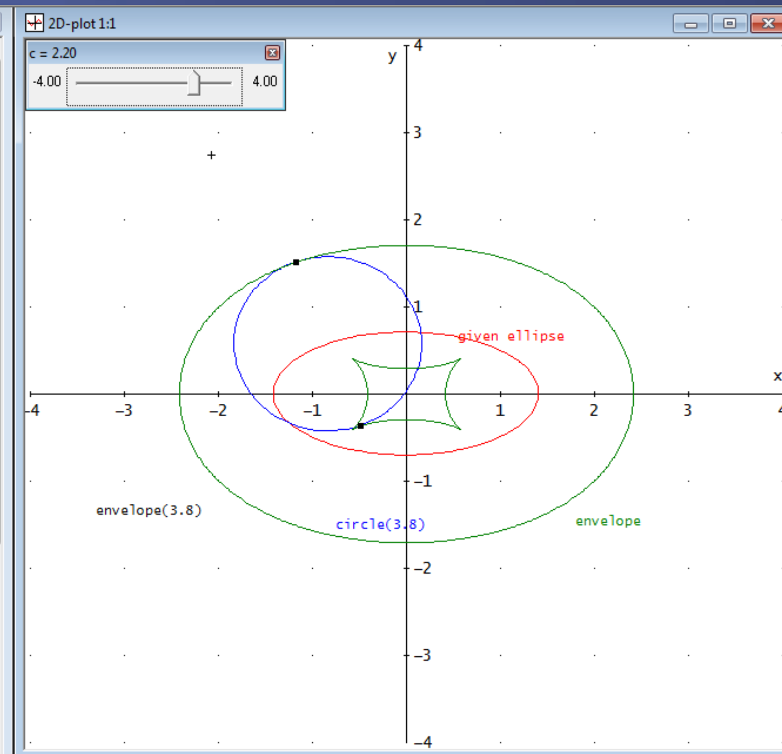
#2: $$\text{circle}(c) := (x - \sqrt{2}\cdot \text{COS}(c))^2 + \left(y - \frac{1}{\sqrt{2}}\cdot \text{SIN}(c)\right)^2 = 1$$

#3: $$\text{envelope}(c) := \text{SOLUTIONS}\left(\text{circle}(c) \wedge \frac{d}{dc}\text{circle}(c), [x, y], \text{Real}\right)$$

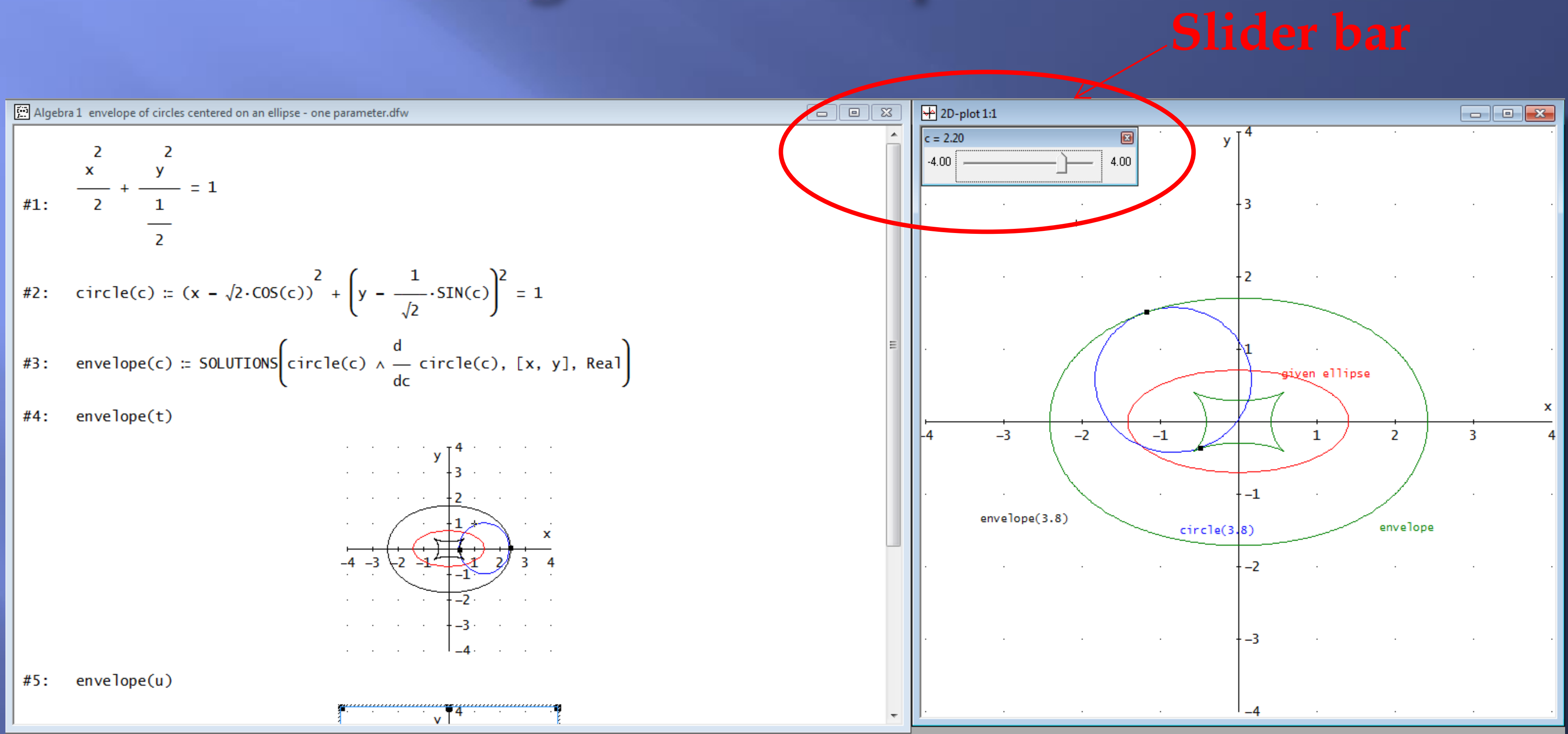
#4: envelope(t)



#5: envelope(u)



Circles whose centers are on a given ellipse



A family which has no envelope

Algebra 1 affine coefficients family of lines - no envelope.dfw

#1: $(m - 1)\cdot x + (m + 1)\cdot y - m = 0$

#2: VECTOR$((m - 1)\cdot x + (m + 1)\cdot y - m = 0, m, -10, 10)$

#3: $\frac{d}{dm}((m - 1)\cdot x + (m + 1)\cdot y - m)$

#4: SOLVE$\left((m - 1)\cdot x + (m + 1)\cdot y - m = 0 \wedge \frac{d}{dm}((m - 1)\cdot x + (m + 1)\cdot y - m), [x, y]\right)$

#5: $x = \frac{1}{2} \wedge y = \frac{1}{2}$

general case of a family of lines with affine coefficients

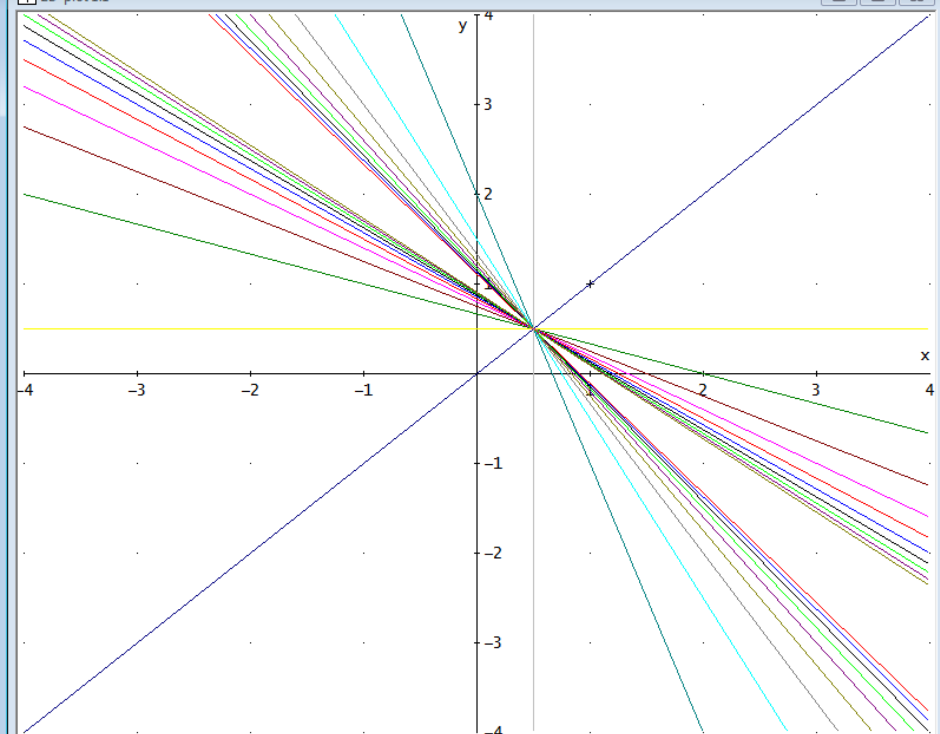
#6: $(a\cdot m + b)\cdot x + (c\cdot m + d)\cdot y + (e\cdot m + f) = 0$

#7: $\frac{d}{dx}((a\cdot m + b)\cdot x + (c\cdot m + d)\cdot y + (e\cdot m + f))$

#8: SOLVE$\left((a\cdot m + b)\cdot x + (c\cdot m + d)\cdot y + (e\cdot m + f) = 0 \wedge \frac{d}{dx}((a\cdot m + b)\cdot x + (c\cdot m + d)\cdot y + (e\cdot m + f)), [x, y]\right)$

#9: false

2D-plot 1:1



General definition of the envelope of a family of plane curves (once again)

Consider a parameterized family *F* of plane curves , dependent on a real parameter *k*. We denote by   an equation for the curves in the family *F*. A plane curve *E* is called **an envelope** of the family *F* if the following properties hold:

(i) every curve is tangent to *E;*

(ii) to every point *M* on *E* is associated a value *k*(*M*) of the parameter *k,* such that is tangent to *E* at the point *M;*

(iii) The function *k*(*M*) is non-constant on every arc of *E*.

Transition towards 3D

- The envelope of a family of surfaces given by an equation $F(x,y,z,c)=0$ is given by the solution of the system of equations:

$$\begin{cases} F(x,y,z,c)=0 \\ \dfrac{\partial F(x,y,z,c)}{\partial c}=0 \end{cases}$$

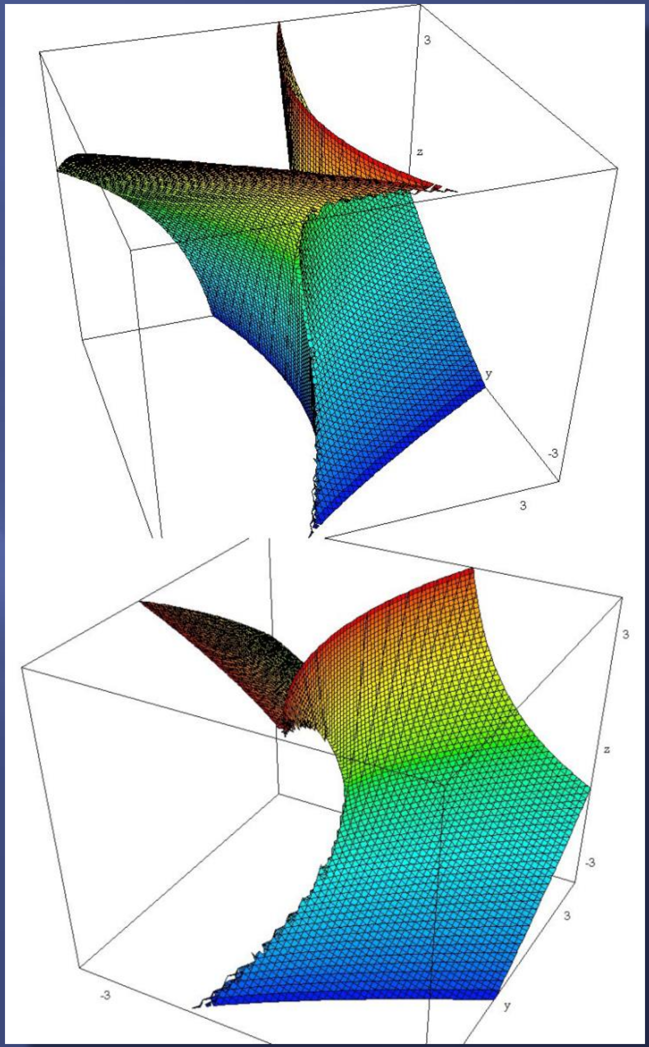
Example

- Let $F(x,y,z,c) = x + cy + c^2 z = c^3.$
- The envelope of the family is given by:

$$\begin{cases} x = c^2(z-2c) \\ y = c(3c-2z) \\ z \in \mathrm{R} \end{cases}$$

- An implicit equation for the envelope:

$$27x^2 + 18xyz + 4xz^3 - 4y^3 - y^2z^2 = 0$$



Here work may be performed by hand

To eliminate c between the first two equations, we solve the equation $y = c(3c - 2z)$ for c, and obtain two solutions: $c = \frac{z - \sqrt{3y + z^2}}{3}$ and $c = \frac{z + \sqrt{3y + z^2}}{3}$.

Then we substitute the two solutions in the equation $x - c^2(z - 2c) = 0$:

$$x - \left(\frac{z - \sqrt{3y + z^2}}{3}\right)^2 \left(z - 2\frac{z - \sqrt{3y + z^2}}{3}\right) = 0$$

$$x - \left(\frac{z + \sqrt{3y + z^2}}{3}\right)^2 \left(z - 2\frac{z + \sqrt{3y + z^2}}{3}\right) = 0$$

We denote the "roots" by x_1 and x_2:

$$x_1 = \left(\frac{z - \sqrt{3y + z^2}}{3}\right)^2 \left(z - 2\frac{z - \sqrt{3y + z^2}}{3}\right)$$

$$x_2 = \left(\frac{z + \sqrt{3y + z^2}}{3}\right)^2 \left(z - 2\frac{z + \sqrt{3y + z^2}}{3}\right)$$

By expanding the equation $27(x - x_1)(x - x_2) = 0$, we obtain an implicit equation for the envelope:

$$27x^2 + 18xyz + 4xz^3 - 4y^3 - y^2z^2 = 0.$$

Here work may be performed by hand

To eliminate c between the first two equations, we solve the equation $y = c(3c - 2z)$ for c, and obtain two solutions: $c = \frac{z - \sqrt{3y + z^2}}{3}$ and $c = \frac{z + \sqrt{3y + z^2}}{3}$.

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$$x - \left(\frac{z + \sqrt{3y + z^2}}{3}\right)^2 \left(z - 2\frac{z + \sqrt{3y + z^2}}{3}\right) = 0$$

BUT NOT ALWAYS!!!

We denote the roots by x_1 and x_2:

$$x_1 = \left(\frac{z - \sqrt{3y + z^2}}{3}\right)^2 \left(z - 2\frac{z - \sqrt{3y + z^2}}{3}\right)$$

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By expanding the equation $27(x - x_1)(x - x_2) = 0$, we obtain an implicit equation for the envelope:

$$27x^2 + 18xyz + 4xz^3 - 4y^3 - y^2z^2 = 0.$$

Remark on the 2D -> 3D transition

	Critical	Non-critical
Algebraic work		■
Numerical work		■
Graphical work	■	

In 3D graphics: it is important to have animated graphs, at least rotating graphs

Problem: implicitization

Solving the system

$$\begin{cases} f(x,y,c)=0 \\ \dfrac{\partial}{\partial c} f(x,y,c)=0 \end{cases}$$

yields a parameterization of the envelope $\begin{cases} x = x(c) \\ y = y(c) \end{cases}$.

Question: when is it possible to find an implicit form for an equation of the envelope?

Answer: not always.

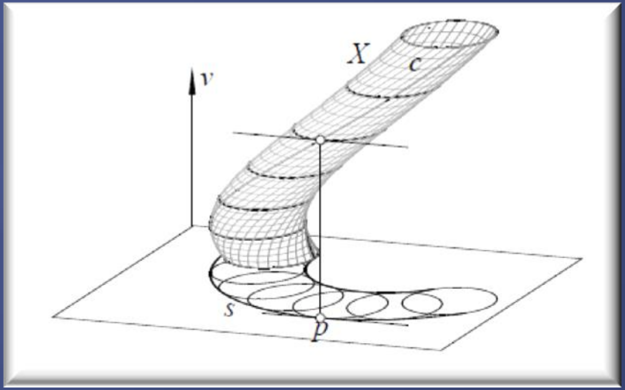
Partial solution: approximate implicitization (T. Schultz and B. Juttler 2010)

Educational aspects

1. **Revival of classical (forgotten?) topics using ICTs**
2. Student engagement
3. The choice of the CAS:
4. Instrumental genesis - registers of representation
5. Extension of the curriculum

Why to revive these topics

- Interesting topic per se (not valid for all students)
- Nowadays provides a blended activity

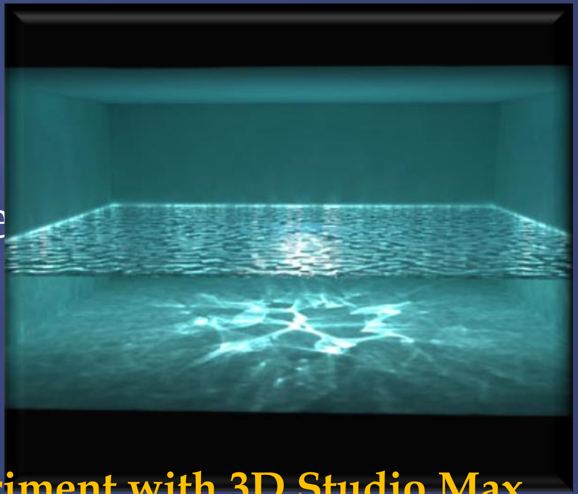


Applications in science and engineering

- Theory of Singularities
- Geometrical Optics: Theory of Caustics, Wave Fronts

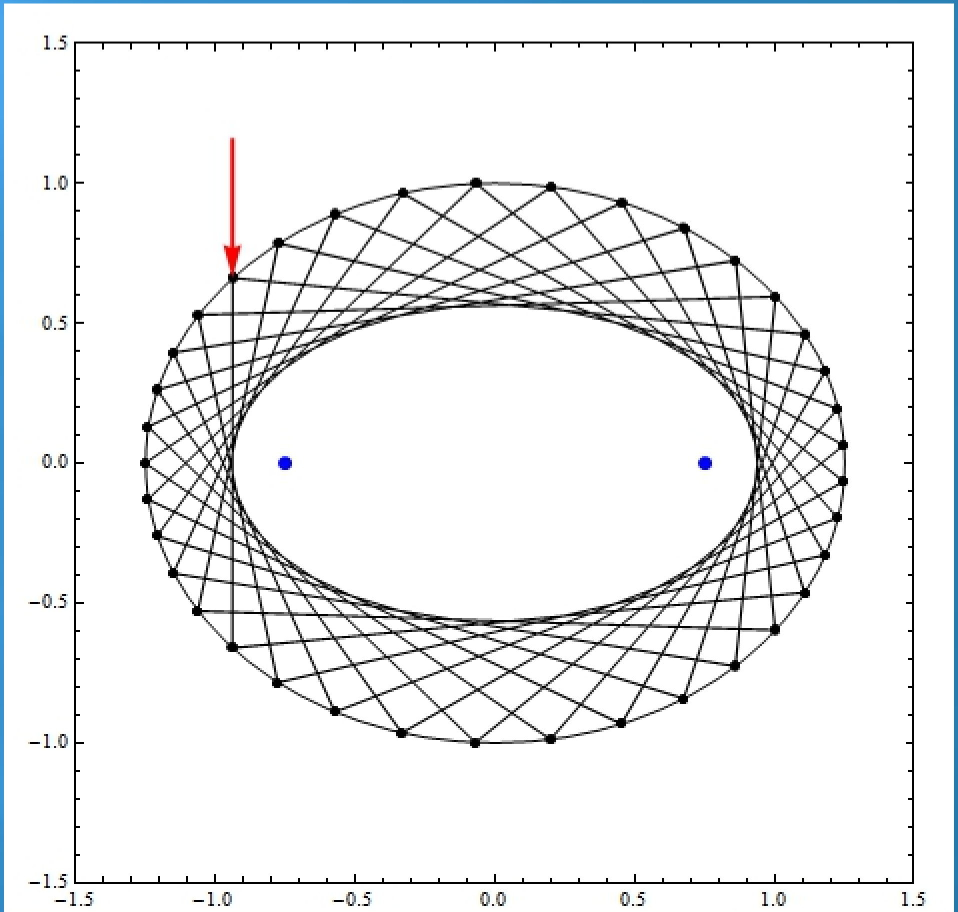
A **caustic** is the envelope of light rays reflected or refracted by a curved surface or object, or the projection of that envelope of rays on another surface. The caustic is a curve (or surface) to which each of the light rays is tangent, defining a boundary of an envelope of rays as a curve of concentrated light. Therefore in the image to the right, the caustics are the bright edges. These shapes often have cusp singularities.

Ref: Arnold 1976, quoted by Capitanio 2002, and Thom 1962).



Experiment with 3D Studio Max using caustics, the reflection of light through water on to a surface.

Optics: Caustic of an ellipse



http://demonstrations.wolfram.com/DynamicBilliardsInEllipse/

Why to revive these topics: Numerous applications in science and engineering

- Robotics and kinematics: rigid body motion in the plane, in 3-space, collision avoidance of robot motion, construction of gears, etc. (Pottman and Peternell 2000).

Educational aspects

1. Revival of classical (forgotten?) topics using ICTs
2. **Student engagement**
3. **The choice of the CAS**
4. Instrumental genesis - registers of representation
5. Extension of the curriculum

Student engagement

- Cognitive engagement
- Behavioral engagement
- Affective engagement
- Problem: In an ICT rich environment, students often are considered as **users rather than learners** (Akbiyik, 2011 - http://www.revistaeducacion.mec.es/re352/re352_08ing.pdf)

Guiding questions for the choice of a CAS

- Depends of age/maturity/CAS literacy of the students
- Menu-driven vs command driven
- Graphics quality
- Animations
- Availability of a slider-bar
- The availability of powerful enough algorithms to solve non linear systems of equations (Gröbner bases, resultants)
- Algorithms for implicitization (idem)
- Sometimes useful: use an additional software
  - v.s. in 3D – Derive and DPGraph

Educational aspects

1. Revival of classical (forgotten?) topics using ICTs
2. Student engagement
3. The choice of the CAS.
4. **Instrumental genesis - registers of representation**
5. Extension of the curriculum

Instrumental genesis

Rabardel & Samurçay, 2001 : the instrument is a mixed entity "made up of both artifact-type components and schematic components … called utilization schemes. This mixed entity is born of both the subject and the object. It is this entity which constitutes the instrument which has a functional value for the subject".

M. Bartolini-Bussi and M.A. Mariotti, 2002: The utilization schemes are progressively elaborated in using the artifact in relation to accomplishing a particular task; thus the instrument is a construction of an individual, it has a psychological character and it is strictly related to the context within which it originates and its development occurs. The elaboration and evolution of the instruments is a long and complex process that Rabardel names *instrumental genesis.*

Instrumental genesis can be articulated into two processes:

- *Instrumentalisation*, concerning the emergence and the evolution of the different components of the artifact, e.g. the progressive recognition of its potentialities and constraints.
- *Instrumentation*, concerning the emergence and development of the utilization schemes.

Switching between different registers of representation

- Registers
  - Algebraic
  - Graphical
  - Numerical
  - Language
  - Etc.

- We refer to works by

  Duval (1995), Robert Speiser and Carolyn Maher (Handbook 2002), Norma Presmeg (ESM 61, 2006), DP (2007), DP and Kidron (2008), etc.

Duval (ESM 61, 2006)

- Two types of transformations on semiotic representations:
  - Treatments: those transformations that can be carried out within the possibilities of a particular system of representation registers.
  - Conversions: those transformations of representations without changing the objects being denoted.
- Each kind of transformation requires different cognitive processes
- These differences must be acknowledged in the teaching and learning of mathematics.

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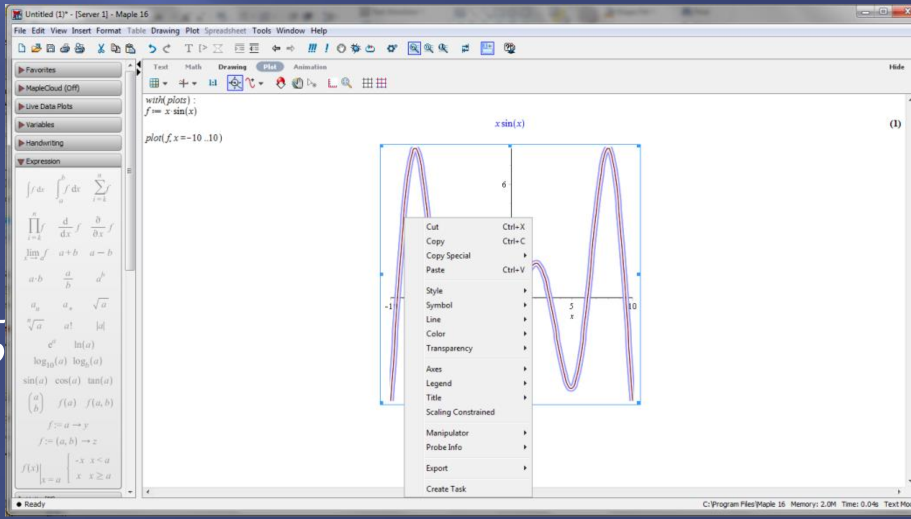
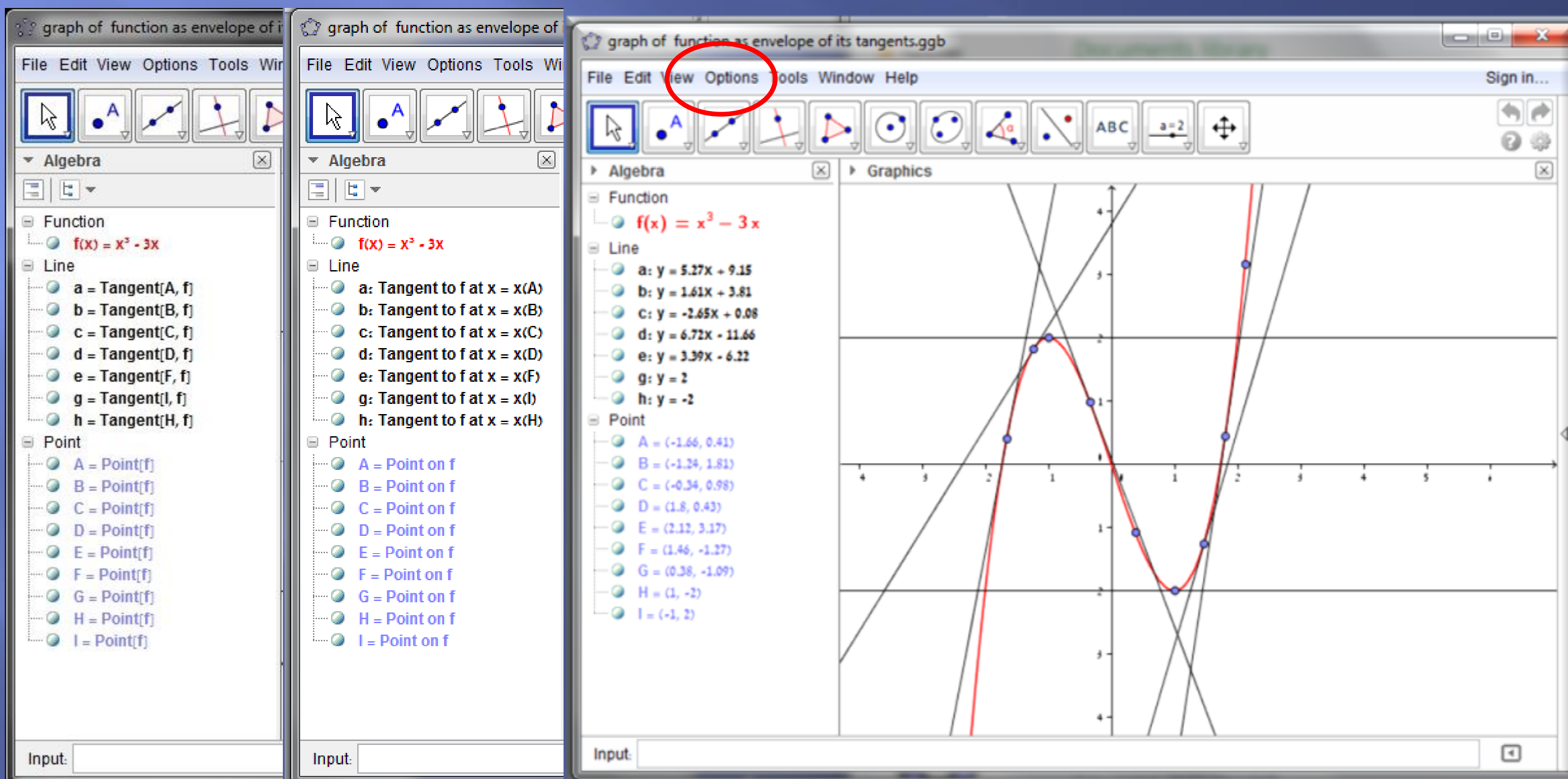
Maple: right mouse click on the graph



Registers: the GeoGebra case

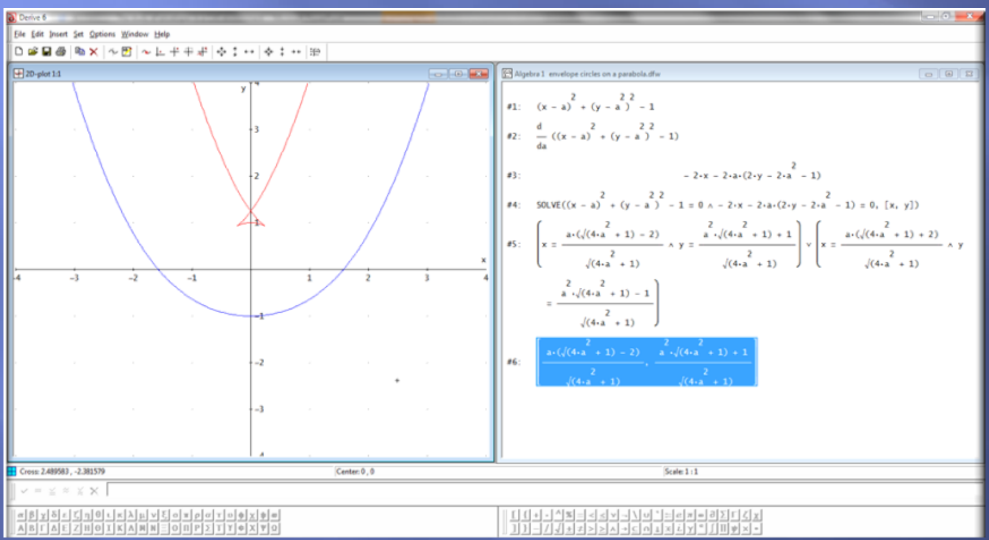
Command Description Value Graphics



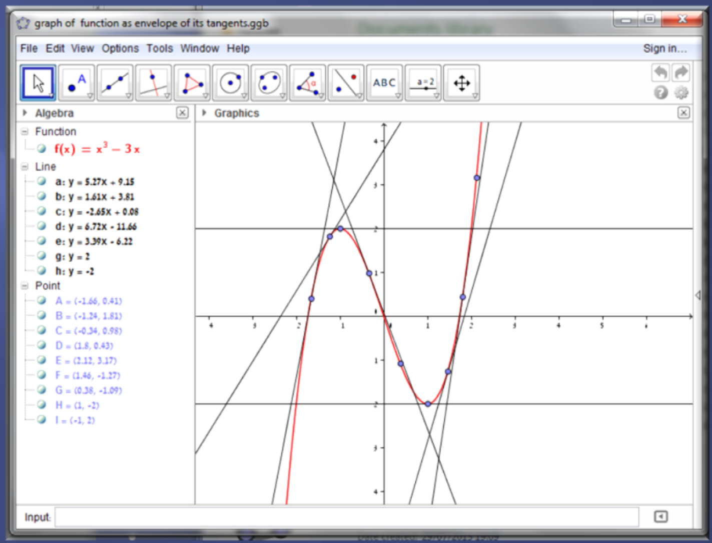
Switching in two reverse directions

From ↓ To →	Algebraic	Numerical	Graphical
Algebraic		V	V
Numerical			V
Graphical		partial	

From ↓ To →	Algebraic	Numerical	Graphical
Algebraic		V	V
Numerical			V
Graphical	V	V	



Derive



GeoGebra

Educational aspects

1. Revival of classical (forgotten?) topics using ICTs
2. Student engagement
3. The choice of the CAS
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5. **Extension of the curriculum**

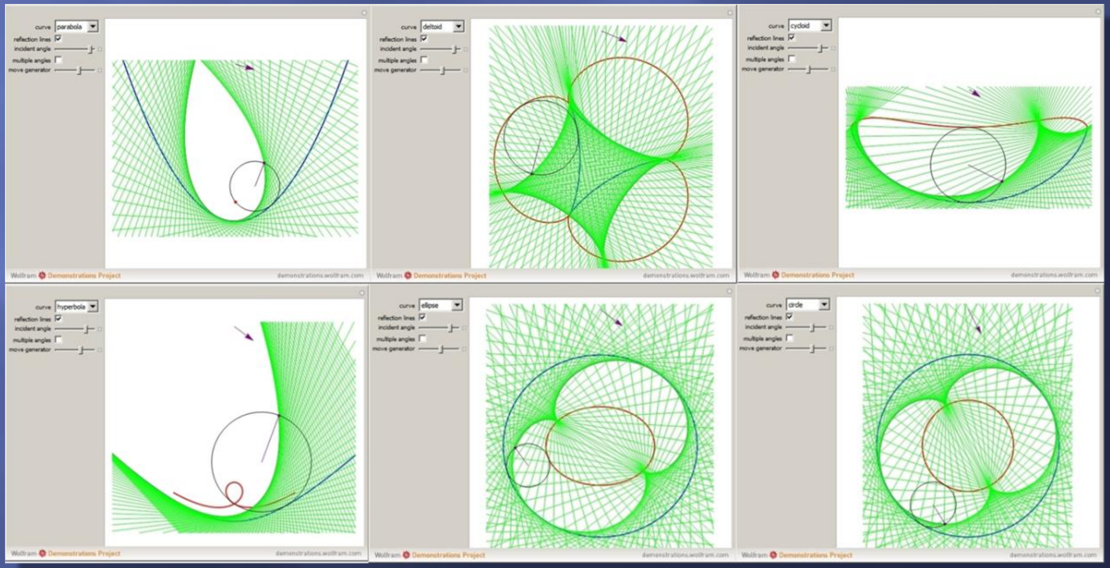
Extension of the curriculum

- DP-Zehavi-Mann:
  - extension of the curriculum with isoptic curves
  - Plane quadrics
  - From conic sections to toric sections

- G. Mann, N. Zehavi and Th. Dana-Picard (2005): *Notes about Reflection, Operative Knowledge, and Execution*, Proceedings of the 4*th* CAME Conference, Roanoke, Virginia, available: http://www.lkl.ac.uk/research/came/events/CAME4/CAME4-topic2-ShortPaper-Mann.pdf.
- N. Zehavi, R. Zaks and Th. Dana-Picard (2006): *Analytic Geometry, Computer Assisted Activities*, Teachers resource e-book, Machshevatika, Dpt of Science Teaching, Weizmann Institute, Rehovot.
- Th. Dana-Picard, G. Mann and N. Zehavi (2006): *New perspectives on conic sections*, Proceedings of TIME-2006 (ACDCA symposium), Dresden, Germany, available: http://rfdz.phnoe.ac.at/fileadmin/Mathematik Uploads/ACDCA/DESTIME2006/DES contribs/Zehavi/Mann Zehavi.pdf
- G. Mann, Th. Dana-Picard and N. Zehavi (2007): *Technological Discourse on CAS-based Operative Knowledge*, International Journal of Technology in Mathematics Education 14 (3), 113-120.

Possible extensions here

- Curvature is generally taught in Calculus courses
- Extension to evolutes
- Study of caustics. In particular in the case of conics.

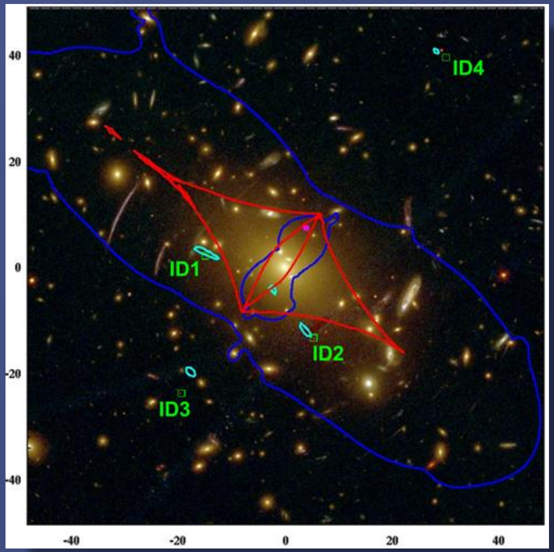


The cosmos in a cup of coffee (Science Daily, April 9th, 2001)

Light rays naturally reflect off a curve like the inside surface of a coffee cup in a curving, ivy leaf pattern that comes to a point in the center and is brightest along its edge.

Mathematicians and physicists call that shape a "cusp curve," and they call the bright edge a "caustic,"... "It happens because a lot of light rays can pile up along curves."

Caustics also show up in gravitational lensing, a phenomenon caused by galaxies so massive that their gravity bends and distorts light from more distant galaxies. "It turns out that their gravity is so powerful that some light rays are also going to pile up along curves," said Petters, a gravitational lensing expert.



"Mother Nature has to be creating these things," Petters said. "It's amazing how what we can see in a coffee cup extends into a mathematical theorem with effects in the cosmos."

DZIĘKUJĘ
THANK YOU
תודה רבה
MERCI
DANKE
GRAZIE
TAK
ありがとう

Dziękuję
Thank you
תודה רבה
Merci
Danke
Grazie
Tak
ありがとう