GINI-Coefficient and GOZINTO-Graph (Workshop)
(Two economic applications of secondary school mathematics)

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Abstract: GINI-Coefficient together with LORENZ-curve and GOZINTO-Graphs are economic applications of secondary school mathematics. They are part of the curriculum in Austria's Colleges for Business Administration (= Handelsakademie).

The GOZINTO-Graph or GOZINTO-Chart is a pictorial representation that shows how the elements required to build a product fit together. This is often the first step for several planning functions especially for material requirements planning, for time and manpower resources planning, for production cost, etc.

This chart gives the input for building the production matrix. Easy to understand and supported by technology easy to perform matrix calculations are leading to the results.
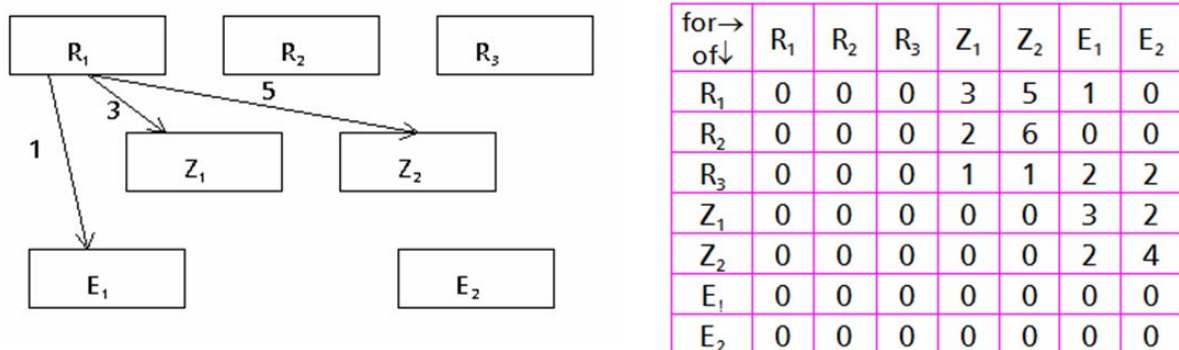
Students can easily be inspired to design own tasks. Examples of students' works will be presented.

The GINI-Coefficient is a means for measuring the income distribution – and other distributions - of a nation's population. It is commonly used as a measure of inequality of income or wealth.

This sounds a little bit complicated but in fact it is an easy to perform application of integration based on the modelling of the so called LORENZ-Curve. We should be happy to find an application of definite integral which is not only calculation of the area between two curves.

GOZINTO-Graph and Technology Matrix (Piece List Problem)

A technical relationship between inputs and the maximum outputs or outcomes of any procedure or process can be represented in form of a graph and/or in form of a matrix.

for→ of↓	R_1	R_2	R_3	Z_1	Z_2	E_1	E_2
R_1	0	0	0	3	5	1	0
R_2	0	0	0	2	6	0	0
R_3	0	0	0	1	1	2	2
Z_1	0	0	0	0	0	3	2
Z_2	0	0	0	0	0	2	4
E_1	0	0	0	0	0	0	0
E_2	0	0	0	0	0	0	0

Accomplish the graph according to the given technology matrix. Two end products need three kinds of raw materials and two kinds of intermediate products.
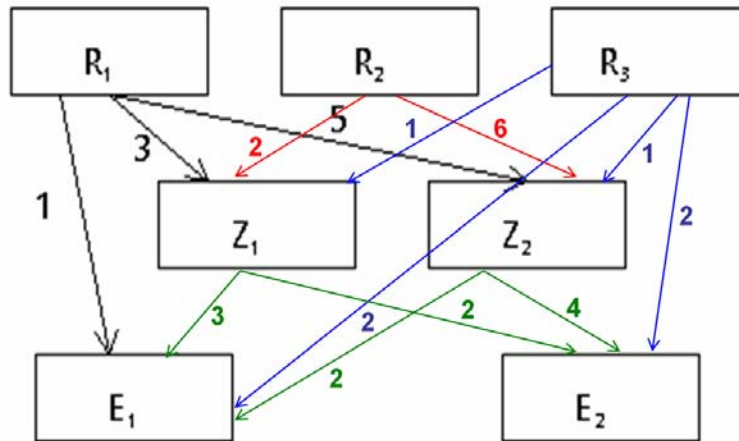
At Flash and Lightning Company

Company Flash & Lightning produces an electronic device which in its basic parts consists of two components A and B which are connected by one resistor and two coils giving the final product. The components A and B are connected by two coils and one resistor forming the device. Component A needs two resistors and one coil and B consists of three coils and one resistor. One coil needs 20 cm of special wire and connection of each coil and each resistor with parts A or B needs additional 2cm of this wire. Additional 10 cm wire are needed for finalising the device.

Set up the GOZINTO-graph preparing the technology matrix.

Calculate the product vectors for the following demands:

a)   5 devices must be produced.

b)   4 devices and 4 replacement parts A and 3 replacement parts B are needed.

c)   6 devices, 8 parts A, 5 parts B, 10 coils and 10 resistors.

d)   All production lists can be calculated in one single step. How can this be performed?

Gozinto Graph:
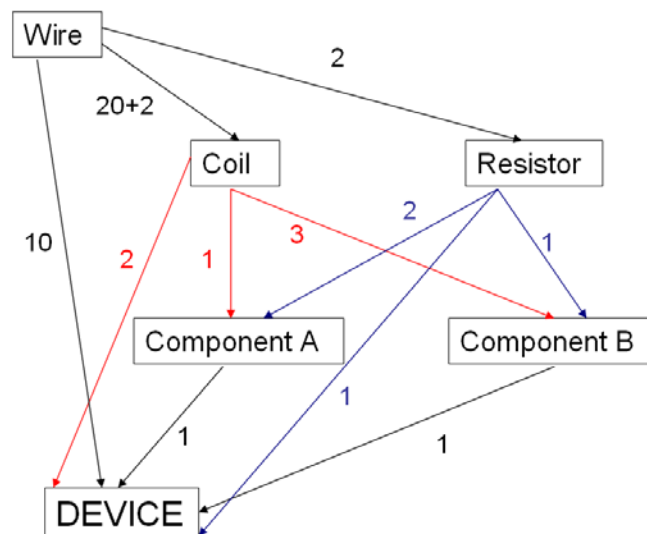
Before working with the technology matrix we can find the product vectors by solving systems of linear equations with unknowns *w, r, c, a, b,* and *d.*

Solutions for questions a) and c):

```
[eq1 := w = 22·c + 10·d + 2·r, eq2 := c = a + 3·b + 2·d]

eq3 := r = 2·a + b + d

[eq4 := a = d, eq5 := b = d, eq6 := d = 5]
```

```
SOLUTIONS([eq1, eq2, eq3, eq4, eq5, eq6], [w, c, r, a, b, d])

[[750, 30, 20, 5, 5, 5]]
```

```
[eq1_ := w = 22·c + 10·d + 2·r, eq2_ := c = a + 3·b + 2·d + 10]

eq3_ := r = 2·a + b + d + 10

[eq4_ := a = d + 8, eq5_ := b = d + 5, eq6_ := d = 6]
```

```
SOLUTIONS([eq1_, eq2_, eq3_, eq4_, eq5_, eq6_], [w, c, r, a, b, d])

[[1688, 69, 55, 14, 11, 6]]
```

This is the technolgy matrix:

$$
\text{FL} := \begin{array}{c} \text{for} \\ \text{of} \end{array}
\begin{array}{cccccc}
w & c & r & a & b & d \\
\begin{bmatrix}
0 & 22 & 2 & 0 & 0 & 10 \\
0 & 0 & 0 & 1 & 3 & 2 \\
0 & 0 & 0 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

As this is a system of linear equations it seems to be very likely that we can solve the problem by means of linear algebra (= matrix calculations):

We show that the system can be represented by the vector-matrix equation from below (*p* = production vector, *D* = technology matrix):

$$p = D \cdot p + d$$

$$p - D \cdot p = d \ \rightarrow \ U \cdot p - D \cdot p = d \ \rightarrow \ (U - D) \cdot p = d$$

$$(U - D)^{-1} \cdot (U - D) \cdot p = (U - D)^{-1} \cdot d$$

$$p = (U - D)^{-1} \cdot d$$

d = demand

Solutions for a), b) and d):

$$(\text{IDENTITY\_MATRIX}(6) - FL)^{-1} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 750 \\ 30 \\ 20 \\ 5 \\ 5 \\ 5 \end{bmatrix} \qquad (\text{IDENTITY\_MATRIX}(6) - FL)^{-1} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 908 \\ 37 \\ 27 \\ 8 \\ 7 \\ 4 \end{bmatrix}$$

$$(\text{IDENTITY\_MATRIX}(6) - FL)^{-1} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 10 \\ 0 & 0 & 10 \\ 0 & 4 & 8 \\ 0 & 3 & 5 \\ 5 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 750 & 908 & 1688 \\ 30 & 37 & 69 \\ 20 & 27 & 55 \\ 5 & 8 & 14 \\ 5 & 7 & 11 \\ 5 & 4 & 6 \end{bmatrix}$$

Additional question: Inspect matrix $(U - D)^{-1}$ and find out its interpretation:

$$(\text{IDENTITY\_MATRIX}(6) - FL)^{-1} = \begin{bmatrix} 1 & 22 & 2 & 26 & 68 & 150 \\ 0 & 1 & 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix D is also called **matrix of direct input coefficients**, matrix $(U - D)^{-1}$ is called **matrix of total input coefficients**. Look at number 68 in row 1 (wire) and column 5 (component B). Producing one unit of B needs 3 coils and 1 resistor, which need $3 \cdot (20 + 2) + 1 \cdot 2 = 68$ units of wire.
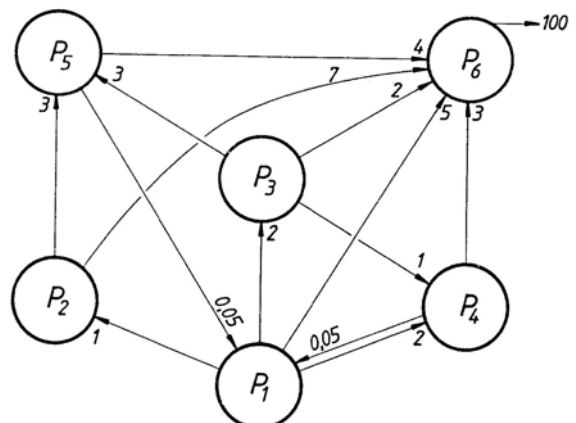
Second Example:

The GOZINTO Graph is provided.

Set up the technology matrix.

Calculate the piece list for an order of 100 pieces P6 plus additional 50 pieces of both P4 and P5.

We show the solution worked out with
TI_NspireCAS:

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$$de := \begin{bmatrix} 0 & 1 & 2 & 2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 1 & 3 & 2 \\ 0.05 & 0 & 0 & 0 & 0 & 3 \\ 0.05 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 1 & 3 & 2 \\ 0.05 & 0 & 0 & 0 & 0 & 3 \\ 0.05 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$demand := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 50 \\ 50 \\ 100 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 50 \\ 50 \\ 100 \end{bmatrix}$$

$$(\text{identity}(6) - de)^{-1} \cdot demand \qquad \begin{bmatrix} 20142.9 \\ 5071.43 \\ 5928.57 \\ 1357.14 \\ 1457.14 \\ 100. \end{bmatrix}$$
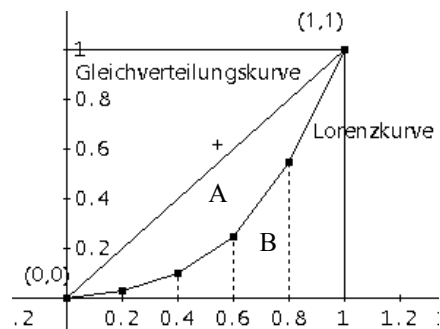
GINI-Coefficient and LORENZ Curve

The GINI coefficient measures the inequality among values of a frequency distribution (for example levels of income). A GINI coefficient of zero expresses perfect equality, where all values are the same (for example, where everyone has the same income). A GINI coefficient of one (or 100%) expresses maximal inequality among values (for example where only one person has all the income).[3][4] However, a value greater than one may occur if some persons represent negative contribution to the total (e.g., have negative income or wealth). For larger groups, values close to or above 1 are very unlikely in practice.

Take the ideal country UTOPIA: All families have the same income. So the "poorest" 10% of the population dispose of 10% of the population income and the poorest 20% dispose of 20% of the population wealth, etc.

In its neighbour country MAMMONIA we can find other circumstance: the poorest 20 can dispose of only 3 % of the population income, the next 20% over 7%, the next 20% of 15%, the next of 30% and the remaining rest of the population can dispose over the remaining rest of the population income. We can represent these facts in a table containing the cumulated percentages and the respective graph.

Sum of the percentage rates of households	Sum of the percentage rates of population income
0	0
20	3
40	10
60	25
80	55
100	100

The GINI coefficient is usually defined mathematically based on the Lorenz curve, which plots the proportion of the total income of the population (y axis) that is cumulatively earned by the bottom x% of the population (see diagram). The line at 45 degrees thus represents perfect equality of incomes. The GINI coefficient can then be thought of as the ratio of the area that lies between the line of equality and the Lorenz curve (marked A in the diagram) over the total area under the line of equality (marked A and B in the diagram); i.e., $G = \dfrac{A}{A+B} = \dfrac{A}{0.5} = 2A$.
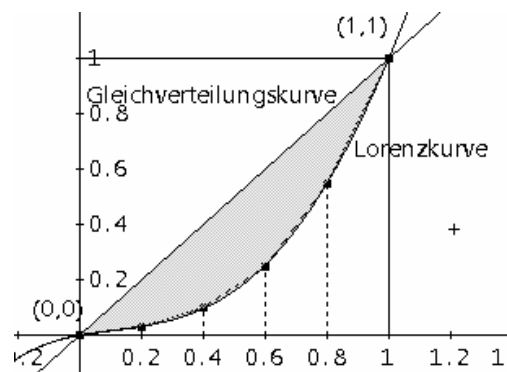
GINI-Koeffizient = 2·Area between $y = x$ and Lorenz curve

$$GK = 1 - \Delta x \cdot \left(2 \cdot \sum_{i=1}^{n} y_i + 1 \right) \quad \text{with } y_i \neq 0,1; \quad GK = 1 - 0.2 \cdot (2 \cdot (0.03 + 0.1 + 0.25 + 0.55) + 1) = 0.428$$

$$\text{lorenz} := FIT\left[x, a \cdot x^4 + b \cdot x^3 + c \cdot x^2 + d \cdot x \right], \begin{bmatrix} 0 & 0 \\ 0.2 & 0.03 \\ 0.4 & 0.1 \\ 0.6 & 0.25 \\ 0.8 & 0.55 \\ 1 & 1 \end{bmatrix}$$

$$2 \cdot \int_0^1 (x - \text{lorenz})\, dx = 0.4443918769$$

$$0 \leq x \leq 1 \wedge x \geq y \geq \text{lorenz}$$

Just recently one can find the GINI coefficient and issues connected with it more and more even in daily newspapers. This is part of an article in journal (left) and a representation of the income distribution of the Austrian population (right). We will use this diagram for calculating the respective GINI-coefficient (which needs the Lorenz curve, too).

mit der Bevölkerung geschlossen: Die Mittelschicht darf reicher werden, dafür wird die Macht der KP nicht infrage gestellt. Was man nicht bedacht hat: Dass die unteren Schichten nicht an den Staat gebunden werden." In China wächst dabei nicht nur die soziale Kluft, sondern auch die Angst vor Unruhen. Der Gini-Koeffizient, der das Maß an Ungleichheit statistisch misst, beträgt offiziell 0,47 und überspringt damit die Marke von 0,4, ab der ein hohes Risiko für Aufstände gilt. Die Südwestuniversität für Finanzen in Chengdu ermittelte in einer privaten Untersuchung gar einen Wert von 0,61. Damit würde China weltweit zur Spitze der ungleichen Länder gehören. „Die offiziellen Zahlen sind seit Jahren unverändert, als ob die Lage gleich geblieben wäre. Aber die Situation hat sich deutlich verschlechtert", sagt die Sinolo-

Economists say that a GINI-coefficient above 0.4 can cause social conflicts between the poor and the wealthy.

EINKOMMEN 2011

Anzahl der Arbeitnehmer in den verschiedenen Einkommensstufen
Bruttojahresbezüge in Euro

	Männer	Frauen	gesamt
bis unter 10.000 €			1.044.002
10.000–19.999			736.724
20.000–29.999			801.692
30.000–39.999			650.471
40.000–49.999			365.900
50.000–59.999			214.814
60.000–69.999			120.749
70.000–79.999			73.330
80.000–89.999			45.778
90.000–99.000			29.928
100.000 und mehr			75.940

KURIER Grafik: Breineder   Quelle: Statistik Austria, Lohnsteuerstatistik 2011

4 160 000 persons totally

Before calculating the GINI-coefficient for the Austrian population let's have a look on the US-GINI coefficient:

Table C. Household money income distributions and Gini Index, USA[60]		
Income bracket (in 2010 adjusted dollars)	% of Population 1979	% of Population 2010
Under $15,000	14.6%	13.7%
$15,000 – $24,999	11.9%	12.0%
$25,000 – $34,999	12.1%	10.9%
$35,000 – $49,999	15.4%	13.9%
$50,000 – $74,999	22.1%	17.7%
$75,000 – $99,999	12.4%	11.4%
$100,000 – $149,999	8.3%	12.1%
$150,000 – $199,999	2.0%	4.5%
$200,000 and over	1.2%	3.9%
Total Households	80,776,000	118,682,000
United States' Gini on pre-tax basis	0.404	0.469

Now to Austria:

First of all we have to calculate the cumulated percentages of the people together with their respective income. Persons and income in 1000.

```
inc := [0, 5, 15, 25, 35, 45, 55, 65, 75, 85, 95, 200]

pers := [0, 1044, 737, 802, 650, 366, 215, 121, 73, 46, 30, 76]

pers_inc := VECTOR(inc ·pers , i, DIM(pers))
                      i    i

pers_inc := [0, 5220, 11055, 20050, 22750, 16470, 11825, 7865, 5475, 3910, 2850, 15200]
```

$$\text{cum\_pers\_perc} := \text{VECTOR}\left(\frac{\sum_{i=1}^{n} \text{pers}_i}{\sum(\text{pers})}, n, 1, \text{DIM(pers)}\right)$$

```
cum_pers_perc = [0, 0.250, 0.428, 0.620, 0.777, 0.865, 0.916, 0.945, 0.963, 0.974, 0.981, 1]
```

$$\text{cum\_pers\_inc\_perc} := \text{VECTOR}\left(\frac{\sum_{i=1}^{n} \text{pers\_inc}_i}{\sum(\text{pers\_inc})}, n, 1, \text{DIM(pers\_inc)}\right)$$

```
cum_pers_inc_perc := [0, 0.0425, 0.132, 0.296, 0.481, 0.615, 0.712, 0.776, 0.820, 0.852, 0.876, 1]
```

This leads us directly to the table of the cumulated percentages of the persons versus their cumulated income. The table can be plotted and gives the Lorenz curve as a polygon from (0,0) to (1,1)

```
inc_distr := [cum_pers_perc, cum_pers_inc_perc]'
```

$$inc\_distr := \begin{bmatrix} 0 & 0 \\ 0.250 & 0.0425 \\ 0.428 & 0.132 \\ 0.620 & 0.296 \\ 0.777 & 0.481 \\ 0.865 & 0.615 \\ 0.916 & 0.712 \\ 0.945 & 0.776 \\ 0.963 & 0.820 \\ 0.974 & 0.852 \\ 0.981 & 0.876 \\ 1 & 1 \end{bmatrix}$$
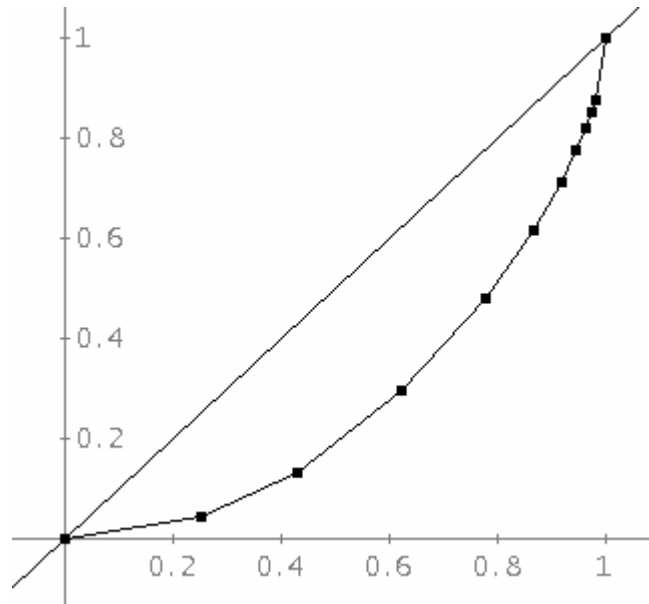
$y = x$

Using regression we try to find a continuous curve approximating the polygon as accurate as possible. And after some tries it turns out that the order six polynomial passing the origin seems to fit best.
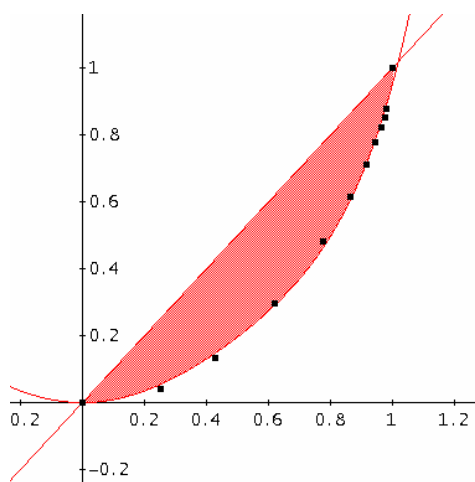
$$FIT\left(\left[x,\ a{\cdot}x^2 + b{\cdot}x + c\right],\ inc\_distr\right) = 1.19{\cdot}x^2 - 0.308{\cdot}x + 0.0204$$

$$FIT\left(\left[x,\ a{\cdot}x^4 + b{\cdot}x^3 + c{\cdot}x^2 + d{\cdot}x\right],\ inc\_distr\right) = 3.37{\cdot}x^4 - 5.90{\cdot}x^3 + 4.05{\cdot}x^2 - 0.570{\cdot}x$$

$$FIT\left(\left[x,\ a{\cdot}x^4 + c{\cdot}x^2\right],\ inc\_distr\right) = 0.360{\cdot}x^4 + 0.569{\cdot}x^2$$

$$FIT\left(\left[x,\ a{\cdot}x^6 + c{\cdot}x^4 + d{\cdot}x^2\right],\ inc\_distr\right) = 0.685{\cdot}x^6 - 0.631{\cdot}x^4 + 0.898{\cdot}x^2$$

$$lc(x) := 0.685{\cdot}x^6 - 0.631{\cdot}x^4 + 0.898{\cdot}x^2$$

$$2{\cdot}\int_0^1 (x - lc(x))\ dx = 0.458$$

AreaBetweenCurves(x, lc(x), x, 0, 1)

The GINI coefficient is 0.46. Various support for the poorer part of the population helps to decrease this social disparity.

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Seven Companies

Seven companies selling similar products share the market in Austria.
Their sales numbers in Mio. EURO in 2011 are given in the table below:

Company	Sales in Mio. EURO
A	20
B	50
C	15
D	15
E	25
F	10
G	20

Discuss the distribution of the sales numbers in Austria.

Work first without using an approximating Lorenz curve and find the GINI-coefficient.

In Economy textbooks one can find this kind of distribution the formula:

$$G = \frac{2\sum\limits_{i=1}^{n} i\,x_i}{n\sum\limits_{i=1}^{n} x_i} - \frac{n+1}{n}.$$
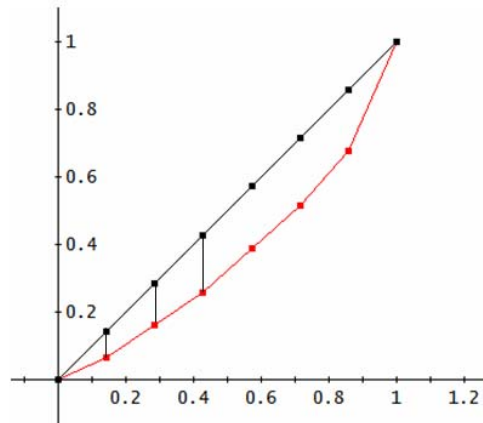
Did you obtain the same result as above?

When will this formula be valid?

Find the GINI-coefficient applying an appropriate approximating curve.

$$
\text{ums\_exakt} := \begin{bmatrix}
0 & 0 \\
\dfrac{1}{7} & \dfrac{10}{155} \\
\dfrac{2}{7} & \dfrac{25}{155} \\
\dfrac{3}{7} & \dfrac{40}{155} \\
\dfrac{4}{7} & \dfrac{60}{155} \\
\dfrac{5}{7} & \dfrac{80}{155} \\
\dfrac{6}{7} & \dfrac{105}{155} \\
1 & 1
\end{bmatrix}
$$

$$\text{gini\_k(ums\_exakt)} = \frac{58}{217}$$

$$\text{gini\_k(ums\_exakt)} = 0.2672811059$$

$$\text{gini\_k}(m) := \sum_{i=1}^{\text{DIM}(m)-1} (m_{i+1,1} - m_{i+1,2} + m_{i,1} - m_{i,2})\cdot(m_{i+1,1} - m_{i,1})$$

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$$\text{umsz} := [10, 15, 15, 20, 20, 25, 50]$$

$$\text{gk\_gl}(m) := \frac{2 \cdot \sum\limits_{i=1}^{\text{DIM}(m)} i \cdot m_i}{\text{DIM}(m) \cdot \sum\limits_{i=1}^{\text{DIM}(m)} m_i} - \frac{\text{DIM}(m) + 1}{\text{DIM}(m)}$$

$$\text{gk\_gl}(\text{umsz}) = \frac{58}{217}$$

$$\text{gk\_gl}(\text{umsz}) = 0.2672811059$$

And now with a Lorenz curve found by regression:

$$\text{FIT}\left(\left[x, \ a \cdot x^2 + b \cdot x + c\right], \ \text{ums\_exakt}\right)$$

$$0.7244623655 \cdot x^2 + 0.2137096774 \cdot x + 0.01747311827$$
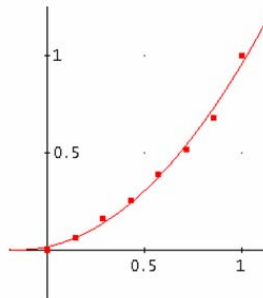
$$2 \cdot \int_0^1 \left(x - \left(0.7244623655 \cdot x^2 + 0.2137096774 \cdot x + 0.01747311827\right)\right) \, dx$$

$$0.2683691756$$

Compare the results!!