The use of DGS and CAS in proving theorems

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The most important question in teaching mathematics

should be

Why?

Telling students things

they can discover on their own,

is a crime.

Introduction

The talk is organized as follows:

- Proving at schools.
- Verification by DGS.
- Visual proofs.
- Classical proofs.
- Proving by CAS.
- Searching for loci.
- Verification in 3D.

Proving at schools

- Proving theorems does not belong to favorite activities in math lessons.
- On the other hand, without proving there is no mathematics.
- How to get over this disproportion?

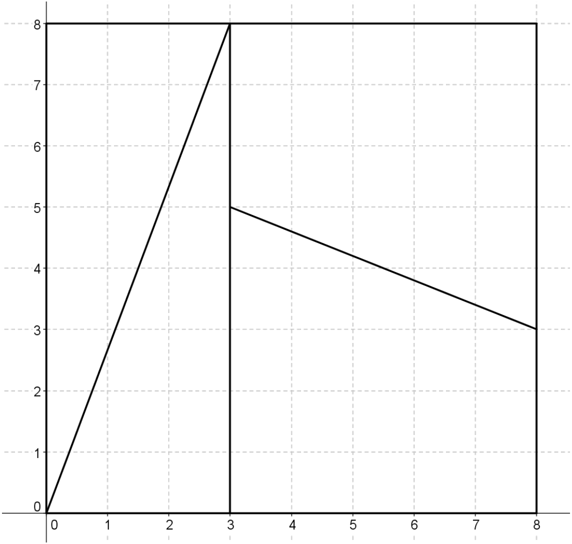
Proving at schools

- Make proving more attractive.
- Persuade students that proofs are necessary.
- Prove such statements we are doubting about.
- Show statements which seem to be true but that are not valid.
- Visualize a proof if possible.
- Show nice proofs.

Make proving more attractive

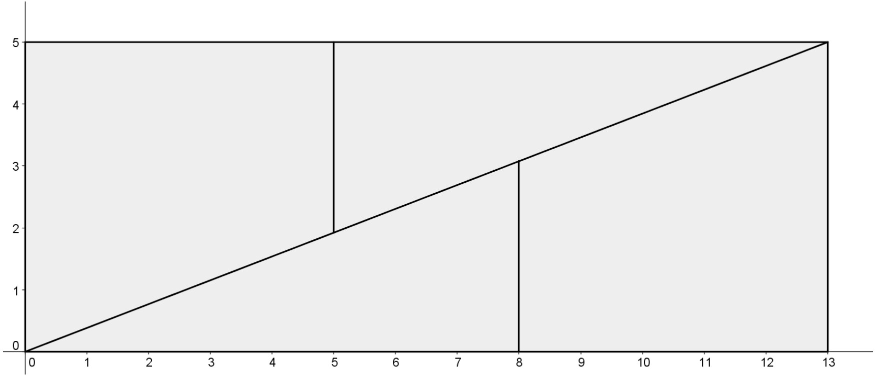
Example

Cut the square into two triangles and two trapezes by the figure



Make proving more attractive

Assemble a rectangular from these 4 pieces



Make proving more attractive

Area of the square equals $8 \times 8 = 64$.

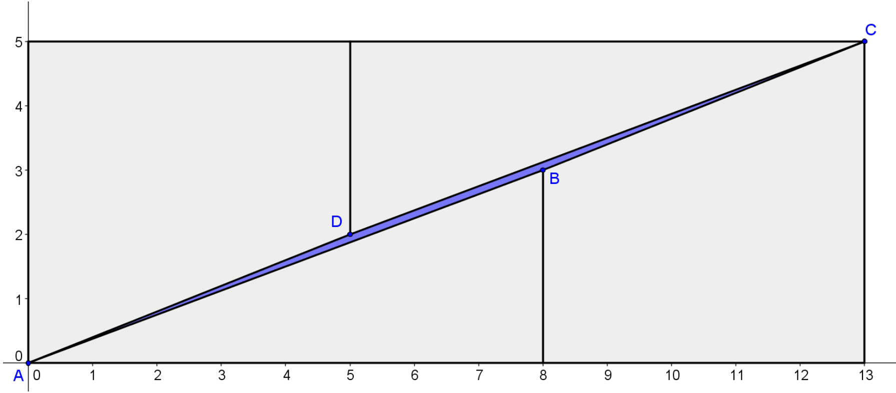
Area of the rectangle equals $5 \times 13 = 65$.

$$64 = 65?$$

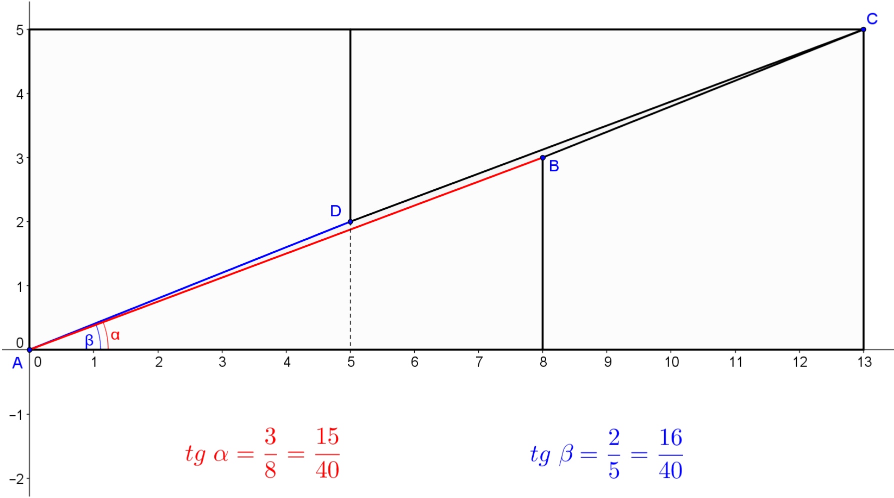
Why?

Make proving more attractive

Area of a parallelogram $ABCD$ equals 1.



Make proving more attractive



$$tg\ \alpha = \frac{3}{8} = \frac{15}{40} \qquad tg\ \beta = \frac{2}{5} = \frac{16}{40}$$

Make proving more attractive

Numbers $5, 8, 13$ which occur in this example are three consecutive members of the well known Fibonacci sequence, where

$$a_{n+1} = a_n + a_{n-1}.$$

For instance the sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ is a Fibonacci sequence. One of its properties is

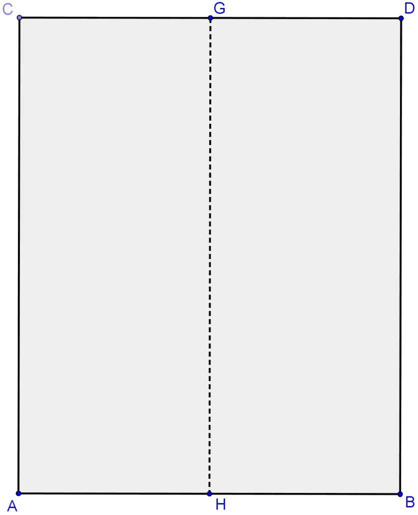
$$a_{n-1} \times a_{n+1} = a_n^2 \pm 1.$$

In our case we have $5 \times 13 = 8^2 + 1$. If we take the next triple $8, 13, 21$ then $8 \times 21 = 13^2 - 1$, etc.

Make proving more attractive

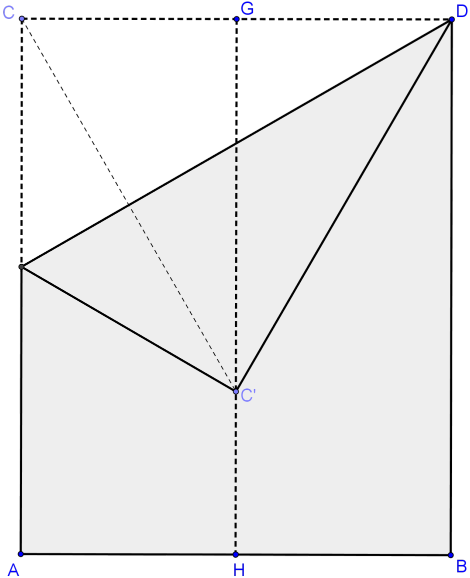
Example

Make an equilateral triangle by paper folding.



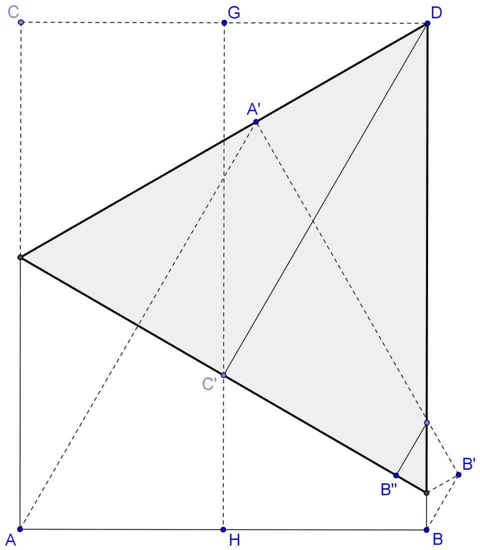
Make proving more attractive

Fold the paper such that C' lies on bisector GH



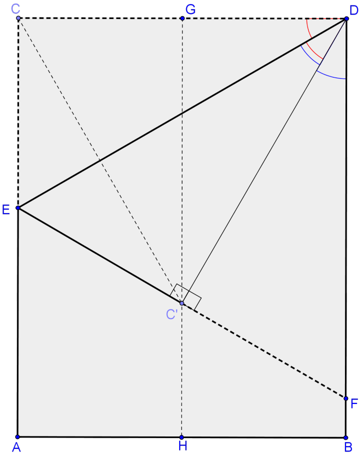
Make proving more attractive

We finish the triangle



Make proving more attractive

Why it is an equilateral triangle?



Δ CDE → Δ C'DE reflection in ED ⇒ Δ CDE ≅ Δ C'DE

∡ CDE = ∡C'DE

|EC'|=|C'F|, GH is a bisector

Δ EFD is isosceles

∡EDC'=∡FDC'

∡CDE=∡C'DE=∡C'DF=30^0

Make proving more attractive

Remarks:

- The use of scissors and a sheet of paper makes mathematics more attractive.
- Manual activity is required.
- First students can measure the angles by a protractor.
- Second students try to find a proof.
- Construction of 3D version — a regular tetrahedron.

Make proving more attractive

- Paper folding is more powerful than Euclidean constructions by ruler and compasses.
- For instance by paper folding we can perform a trisection of an arbitrary angle.
- Introduce paper folding into math lessons.
- We should always ask: Why?

Verification in DGS

- Verification in DGS: A statement is numerically checked by DGS in infinitely many instances.
- Verification in DGS can be considered as the first step of proving.
- Verification in DGS can replace a proof in lower classes at schools.
- Verification in DGS is not a proof.

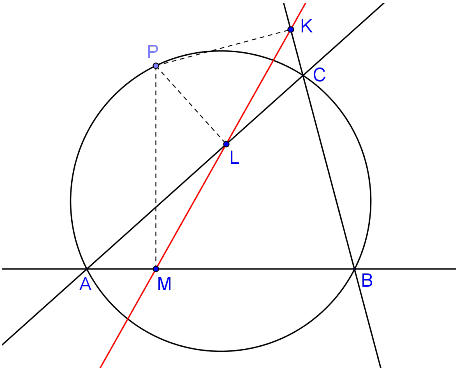
Verification in DGS

- There is a high probability that a statement which is verified in DGS is valid.
- Verification in DGS enables stating conjectures.
- Verification in DGS motivates students.

Verification in DGS

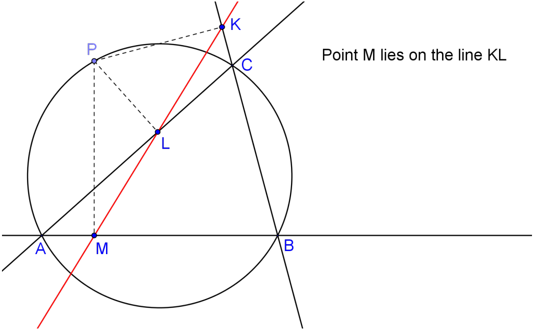
Example

(Simson–Wallace): Let P be a point of the circumcircle of $\triangle ABC$. Then orthogonal projections K, L, M of P onto the sides of ABC are collinear.



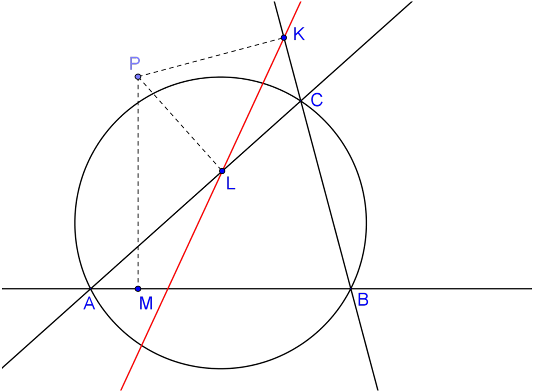
Verification in DGS

- We show that M lies on the line KL.
- Clicking in GeoGebra on "Relation between Two Objects" we get the answer: *Point M lies on the line KL*.
- If P moves along the circumcircle, we still see this text.



Verification in DGS

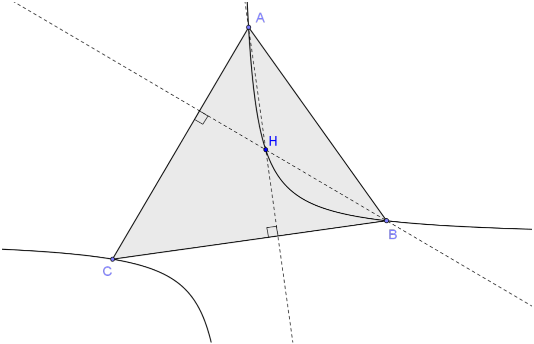
If we detach P from the circumcircle the text disappears. We see that points K, L, M are not collinear.



Verification in DGS

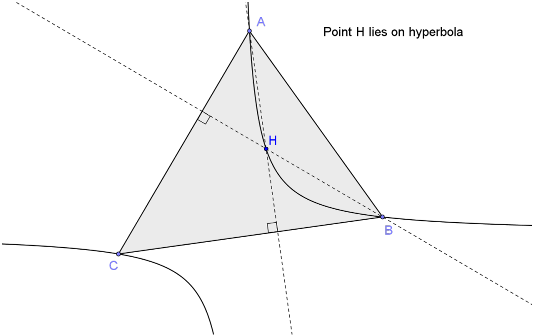
Example

Given $\triangle ABC$ with vertices on an equilateral hyperbola. Then the orthocenter H of $\triangle ABC$ lies on the hyperbola as well.



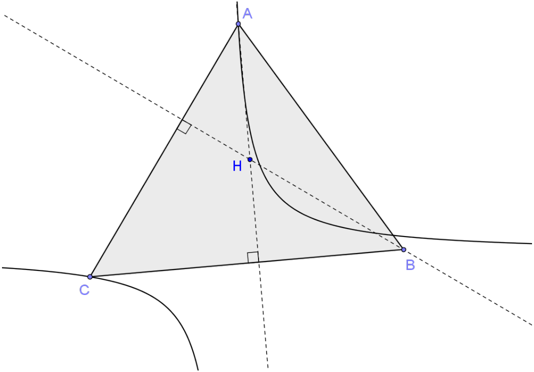
Verification in DGS

- We ask whether the orthocenter H lies on hyperbola.
- Using the window "Relation between two objects", we get the answer *Point H lies on hyperbola*.
- If we move the vertices A, B, C along the hyperbola, we still receive the same answer.



Verification in DGS

- We can persuade students that the text works right (and we do not cheat them).
- Using the window "Detach" we detach the vertex B from the hyperbola, and the text disappears.
- We see that the orthocenter H does not lie on the hyperbola.



Verification in DGS

In the last example we investigated one of properties of an equilateral hyperbola.

- Equilateral hyperbola has in a certain sense the same importance as a circle.
- It canonical equation is $x^2 - y^2 = 1$, whereas the equation of a circle is $x^2 + y^2 = 1$.
- That is why an equilateral hyperbola has many unique properties. One of them was given above.
- Everybody knows that an equilateral hyperbola is a graph of indirect proportion or a graph of linear fractional function.

Verification in DGS

- We can also verify geometric inequalities.
- It is clear that we are not able to check all positions of the triangle ABC.
- Mathematical proof is necessary.
- Let us see the next example.

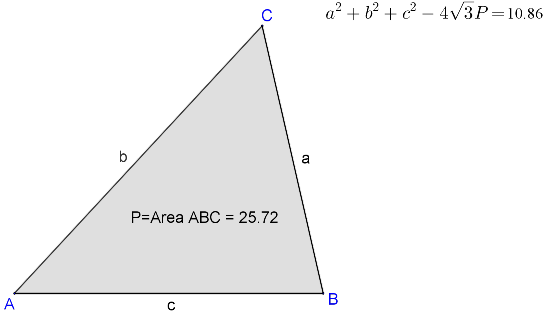
Verification in DGS

Example

Given a triangle with side lengths a, b, c and area P. Then

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}P$$

with equality for an equilateral triangle.



Visual proofs

Sometimes, to prove a statement, we can use a visual proof.

- Visual proofs are closely connected with DGS.
- Visual proof is a proof which follows from the figure.
- Visual proofs are the best proofs, especially at schools.
- In the following we demonstrate a visual proof of the previous inequality.

Visual proofs

Inequality

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}P$$

can be written in the form

$$\frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} + \frac{c^2\sqrt{3}}{4} \geq 3P.$$

Realize that

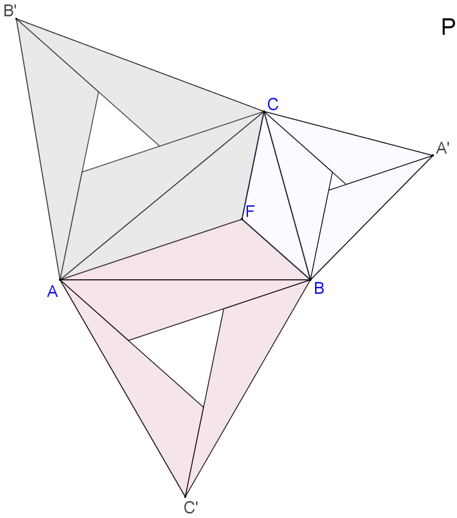
$$P_a = \frac{a^2\sqrt{3}}{4}, \quad P_b = \frac{b^2\sqrt{3}}{4} \quad \text{and} \quad P_c = \frac{c^2\sqrt{3}}{4}$$

are areas of equilateral triangles with side lengths a, b and c.

Then we get

$$P_a + P_b + P_c \quad \geq \quad 3P.$$

Visual proofs



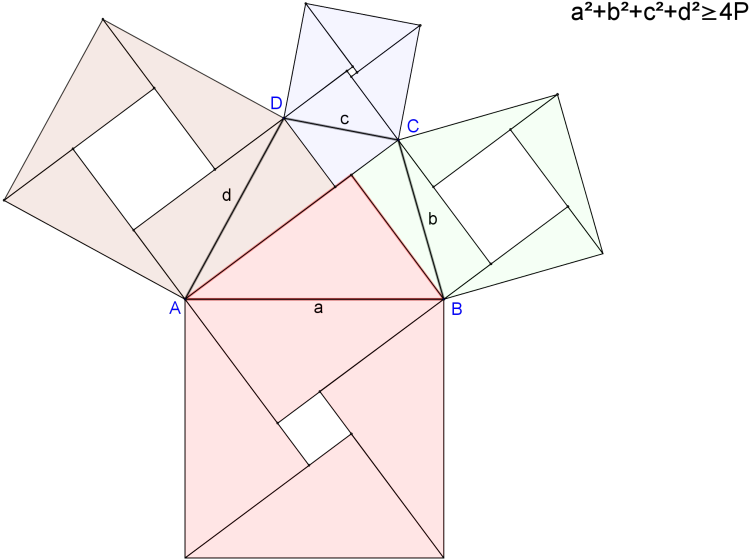
$$a^2 + b^2 + c^2 \geq 4\sqrt{3}P$$

<=>

P(A'BC)+P(AB'C)+P(ABC') ≥ 3P

Visual proofs

The same inequality for a quadrilateral *ABCD*



Visual proofs

Two previous inequalities were special cases of the inequality

$$a_1^2 + a_2^2 + ... + a_n^2 \quad \geq \quad 4tg\frac{\pi}{n}P,$$

where $a_1, a_2, \ldots, a_n$ are the side lengths of a plane n-gon with area P.
The equality is attained iff the n-gon is regular.

For $n = 3$ we get

$$a^2 + b^2 + c^2 \ \geq \ 4\sqrt{3}P,$$

for $n = 4$ we get

$$a^2 + b^2 + c^2 + c^2 \ \geq \ 4P.$$

Visual proofs

- Visual proofs in connection with DGS represent a very strong tool for proving at schools.
- Visual proofs are very convincing.
- To find a visual proof of a statement is not easy. It requires deep knowledge and experience.
- Use visual proofs at schools whenever it is possible.

Visual proofs

See:

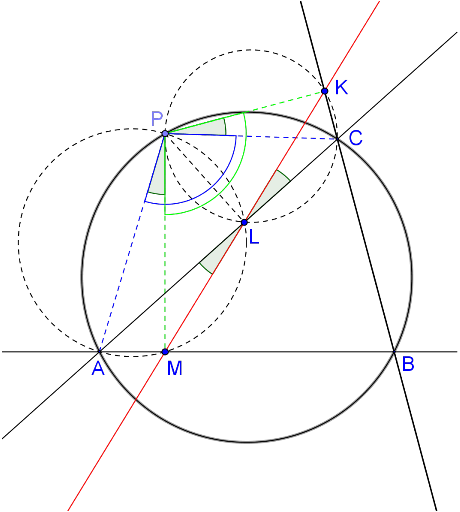
- R. B. Nelsen: Proofs without words I, II.
- C. Alsina, R. B. Nelsen: Charming proofs: A journey into elegant mathematics.
- B. Polster: Q.E.D. Beauty in mathematical proof.

Classical proofs

- By DGS we recognize relations between geometric objects.
- By changing positions of objects in DGS we can analyze the problem.
- This could help us to find a way to a classical (readable) proof.
- There exists software on how to find relations between subjects, e.g. OK Geometry by prof. Magajna, ...
- Many researchers explore ways how to find readable proofs by computer.

Classical proofs

Classical proof of the Simson–Wallace theorem:



K,L,M collinear ⇔

∡ALM=∡CLK ⇔ ∡APM=∡CPK

as ALMP and CLPK are cyclic quadrilaterals

ABCP and MBKP are cyclic quadrilaterals ⇒

∡APC=∡MPK since sum of opposite angels equals 180^0

Thus

∡APM=∡CPK Q.E.D.

Classical proofs

- We needed a key idea to prove classically the Simson–Wallace theorem.
- That is why proving statements is difficult (not only at schools).
- What to do if we do not have a key idea .....
- Therefore scientists try to find an automated way of proving.

Classical proofs

- Classical proof requires a deep insight into the problem.
- If we do not know a classical proof then a computer proof comes into consideration.
- We should give priority to classical proving over computer proving.

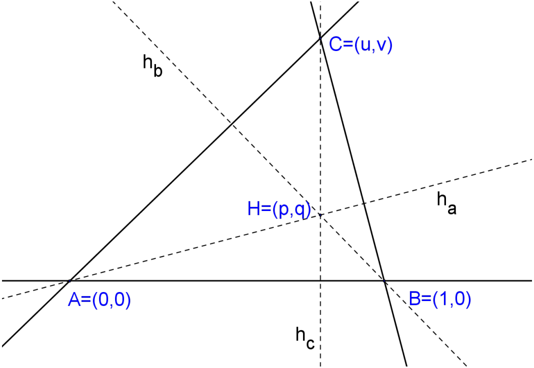
Proving by CAS

- Theory of automated geometry theorem proving.
- Proving theorems.
- Discovering theorems.
- Based on results of commutative algebra in last 40 years.
- Two basic methods — Groebner bases method and Wu–Ritt method.
- (Radical) ideal membership theorem, Buchberger's algorithm.

Proving by CAS

Example

Three heights of a triangle are concurrent.



Introduce a rectangular coordinate system such that $A = (0, 0)$, $B = (1, 0)$, $C = (u, v)$ and $H = (p, q)$.

Proving by CAS

We'll show that

$$\text{if} \quad H \in h_a \cap h_b \quad \text{then} \quad H \in h_c.$$

$$H \in h_a \Leftrightarrow (H - A) \cdot (C - B) = 0 \Leftrightarrow h_1 := p(u-1) + qv = 0.$$

$$H \in h_b \Leftrightarrow (H - B) \cdot (C - A) = 0 \Leftrightarrow h_2 := (p-1)u + qv = 0.$$

$$H \in h_c \Leftrightarrow (H - C) \cdot (B - A) = 0 \Leftrightarrow h_3 := p - u = 0.$$

Proving by CAS

How to show that from

$$p(u-1)+qv=0 \quad \text{and} \quad (p-1)u+qv=0$$

the relation $p-u=0$ follows?

It suffices to write

$$1\cdot[p(u-1)+qv]+(-1)\cdot[(p-1)u+qv]=p-u$$

or

$$1\cdot h_1+(-1)\cdot h_2=h_3.$$

If $h_1=0$ and $h_2=0$ then from the last relation we get $h_3=0$.

Q.E.D.

Proving by CAS

All procedure can be done automatically by computer. In the software CoCoA[1] we enter

```
Use R::=Q[u,v,p,q];
I:=Ideal(p(u-1)+qv,(p-1)u+qv);
NF(p-u,I);
```

and get NF $= 0$.

This means that the conclusion polynomial $p - u$ can be expressed as a linear combination of hypotheses polynomials $p(u-1) + qv$ and $(p-1)u + qv$.

[1] Program CoCoA is freely distributed at http://cocoa.dima.unige.it

Proving by CAS

If we want to know the coefficients of this linear combination, we enter

```
Use R::=Q[u,v,p,q];
I:=Ideal(p(u-1)+qv,(p-1)u+qv);
GenRepr(p-u,I);
```

and get the answer $[1, -1]$. This means that

$$1 \cdot h_1 + (-1)h_2 = h_3.$$

Proving by CAS

To prove the last theorem in a math lesson:

- First verify the theorem in DGS. Show that $H = h_a \cap h_b$ belongs to h_c.
- Second try to prove the theorem classically.
- If we find a classical proof then we can omit the computer proof.

Proving by CAS

Proving by CAS consists of the following steps:

- Introduction of a suitable coordinate system (if necessary).
- Translation of geometric relations into algebraic equations and inequations.
- Expression of a conclusion polynomial in the form of a linear combination of hypotheses polynomials. If is not possible then go on by:
- Searching for subsidiary conditions (cases of degeneracy, objects are not defined, etc.) and adding them to the original hypotheses.

Proving by CAS

- Now the process repeats. Go on by:
- Expression of a conclusion polynomial in the form of a linear combination of hypotheses polynomials plus subsidiary conditions.
- The theory is not complete (in real geometry), i.e. it can happen that we can not decide whether the statement is true or not.
- Some of these steps are still not solved in the theory of automated geometry theorem proving, e.g. translation of subsidiary conditions back into geometry.

Proving by CAS

How to use proving by CAS in math lessons?

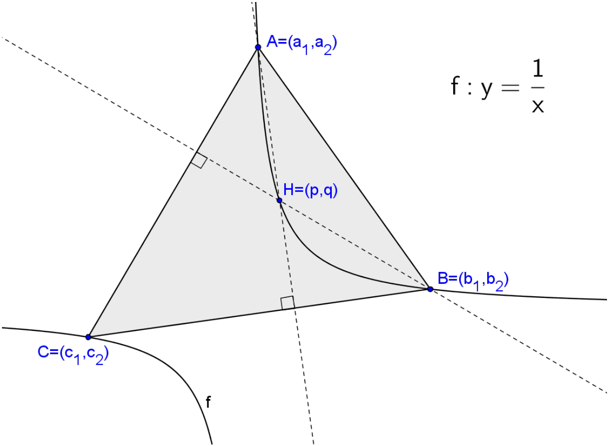
- It seems that automated proof is a "button" proof, i.e. press a button and receive the answer "yes" or "not".
- From pedagogical point of view "Button proofs" are not suitable at schools.
- But the situation is more complicated. It is not easy to do an automated proof of a statement, because usually human interaction is needed.
- By my experience from the Faculty of Education, University of South Bohemia, only best students are able to produce an automated proof without help.

Proving by CAS

Let us see the automated proof of the theorem which we verified in DGS before:

Example revisited

Given $\triangle ABC$ with vertices on an equilateral hyperbola. Then the orthocenter H of $\triangle ABC$ lies on the hyperbola as well.



Proving by CAS

- Introduce a coordinate system such that $A = (a_1, a_2)$, $b = (b_1, b_2)$, $C = (c_1, c_2)$ and $H = (p, q)$.
- Describe geometric objects and relation between them:

$A \in f \Leftrightarrow a_1 a_2 - 1 = 0,$

$B \in f \Leftrightarrow b_1 b_2 - 1 = 0,$

$C \in f \Leftrightarrow c_1 c_2 - 1 = 0,$

$(H - A) \perp (B - C) \Leftrightarrow (p - a_1, q - a_2) \cdot (b_1 - c_1, b_2 - c_2) = 0$

$(H - B) \perp (C - A) \Leftrightarrow (p - b_1, q - b_2) \cdot (c_1 - a_1, c_2 - a_2) = 0.$

We want to show that

$H \in f \Leftrightarrow pq - 1 = 0.$

Proving by CAS

We enter

```
Use R::=Q[a[1..2],b[1..2],c[1..2],p,q];
I:=Ideal(a[1]a[2]-1,b[1]b[2]-1,c[1]c[2]-1,
(p-a[1])(b[1]-c[1]),(q-a[2])(b[2]-c[2]),
(p-b[1])(c[1]-a[1]),(q-b[2])(c[2]-a[2]));
NF(pq-1,I);
```

and get $NF \neq 0$.

But if we add a subsidiary condition $(b_1 - c_1)(b_2 - c_2) \neq 0$ to the ideal I then $NF = 0$ and the theorem is proved.

Realize that $(b_1 - c_1)(b_2 - c_2) \neq 0$ geometrically means that $B \neq C$ which is acceptable since otherwise the triangle ABC degenerates.

Proving by CAS

Then we enter

```
Use R::=Q[a[1..2],b[1..2],c[1..2],p,q,t];
J:=Ideal(a[1]a[2]-1,b[1]b[2]-1,c[1]c[2]-1,
(p-a[1])(b[1]-c[1]),(q-a[2])(b[2]-c[2]),
(p-b[1])(c[1]-a[1]),(q-b[2])(c[2]-a[2]),
(b[1]-c[1])(b[2]-c[2])t-1);
NF(pq-1,J);
```

and get `NF=0`.

The theorem is proved.

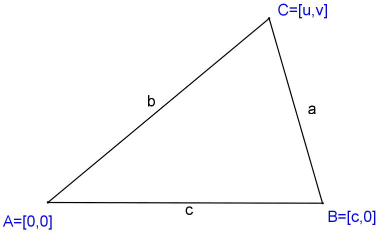
Proving by CAS

Example revisited

Given a triangle with side lengths a, b, c and area P. Then

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}P$$

with equality for an equilateral triangle.



We introduce a coordinate system such that $A = (0, 0)$, $B = (c, 0)$, $C = (u, v)$. Then

Proving by CAS

$$a = |BC| \Leftrightarrow a^2 = (u - c)^2 + v^2,$$

$$b = |CA| \Leftrightarrow b^2 = u^2 + v^2,$$

$$P = \text{area of } ABC \Leftrightarrow P = 1/2cv.$$

We'll write the left side in terms of coordinates

$$a^2 + b^2 + c^2 - 4\sqrt{3}P = (u - c)^2 + 2v^2 + u^2 + c^2 - 2\sqrt{3}cv$$

which can be expressed as the sum of squares

$$a^2 + b^2 + c^2 - 4\sqrt{3}P = 2(u - c/2)^2 + 2(v - c\sqrt{3}/2)^2 \geq 0 \ .$$

The equality is attained iff $u = c/2$ and $v = c\sqrt{3}/2$, i.e, ABC is equilateral.

Proving by CAS

- The use of CAS caused a revolution in proving theorems.
- Hundreds of theorems were proved and even discovered.
- There are still many unsolved problems which await their solution.
- Efficiency of proving by CAS depends both on the power of computers and on algorithms based on the level of mathematical knowledge.
- Almost 40 years elapsed since 1976 when the first computer proof was done - Four colour problem.

Searching for loci

- Searching for loci of points belongs to the most difficult parts of geometry curricula at all school levels.
- New technology tools facilitate this problem considerably.
- Both DGS and CAS are used.
- The valuable topic for students.

Searching for loci

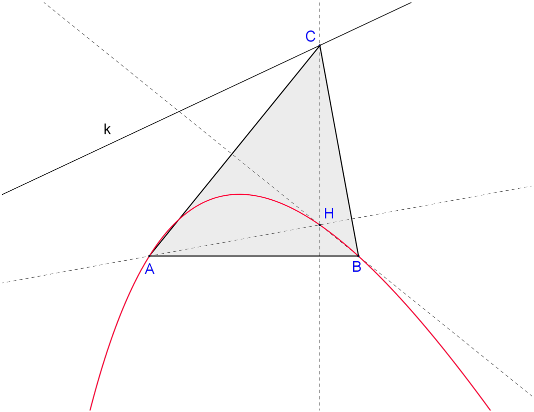
By searching for loci we keep with students the following rules:

- First demonstrate the problem with DGS and construct some points of the searched locus.
- On the base of the previous step try to guess the locus.
- Then use the window "Locus" (Geogebra, Cabri,...) to draw the locus.
- Use CAS to determine the locus equation.

Searching for loci

Example

Let ABC be a triangle with a base AB and a vertex C on a given line k. Find the locus of the orthocenter H of ABC when C moves along the line k.



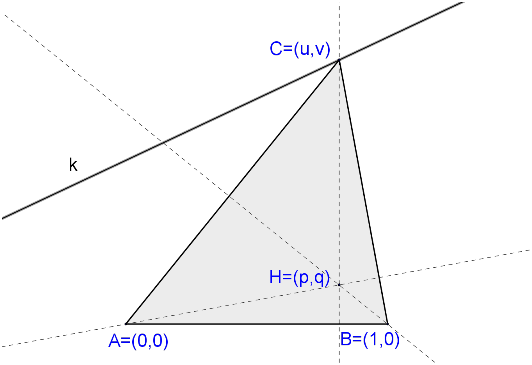
Searching for loci

What is it?

- Some students say: It is a parabola.
- Another students say: It is a hyperbola.
- Or it is neither parabola nor hyperbola?
- What is the solution?
- We'll search for the locus equation.

Searching for loci

Introduce a coordinate system such that $A = (0, 0)$, $B = (1, 0)$,
$C = (u, v)$, $H = (p, q)$
and let k be an arbitrary line $k : ax + by + c = 0$.



Searching for loci

For the intersection H it holds:

$$(H - C) \perp (B - A) \Leftrightarrow h_1 : (p - u, q - v) \cdot (1, 0) = 0,$$

$$(H - A) \perp (C - B) \Leftrightarrow h_2 : (p, q) \cdot (u - 1, v) = 0.$$

Further

$$C \in k \Leftrightarrow h_3 : au + bv + c = 0.$$

Searching for loci

We get the system of three equations $h_1 = 0$, $h_2 = 0$, $h_3 = 0$ in variables u, v, p, q, a, b, c.

To find the locus of $H = (p, q)$ we eliminate variables u, v in the ideal $I = (h_1, h_2, h_3)$ to obtain a relation in p, q which depends only on a, b, c. We enter

```
Use R::=Q[a,b,c,u,v,p,q];
I:=Ideal(au+bv+c,p-u,(u-1)p+vq);
Elim(u..v,I);
```

and get the equation

$$\kappa : bp^2 - apq - bp - cq = 0.$$

Searching for loci

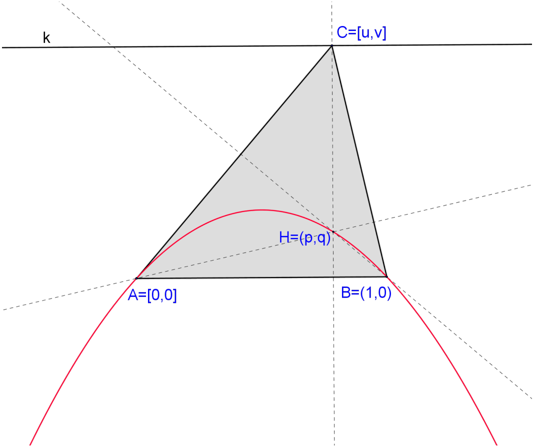
- Suppose that $(a, b) \neq (0, 0)$ since in this case the line k is not defined. Then $\kappa = 0$ is the equation of a conic.

- The cases $k = h_{AB}$, $k = AC$, or $k = BC$ lead to singular conics which consist of two intersecting lines which are not depicted.

- Considering regular conics we get two cases:

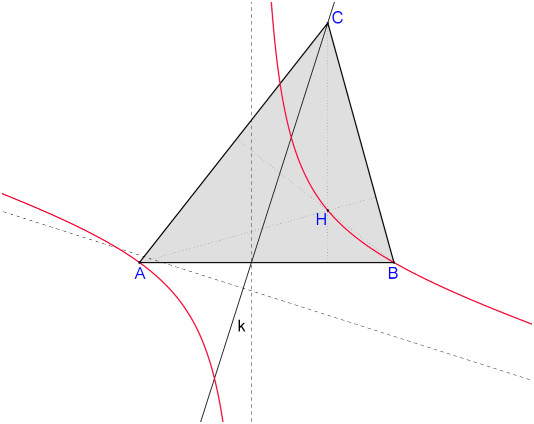
Locus equations

1. If $k \parallel AB$ the locus is a parabola with the vertex $(1/2, -b/(4c))$ and a parameter $|c/(2b)|$.



Searching for loci

2. If $k \nparallel AB$ we obtain a hyperbola centered at $(-c/a, -b(a+2c)/a^2)$ with one asymptote perpendicular to AB and the second asymptote perpendicular to the line k.



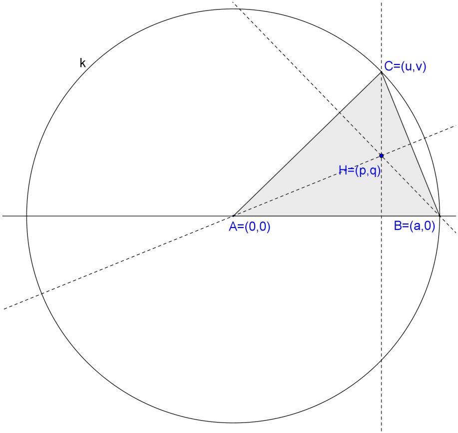
Searching for loci

- The locus above was found by algebraic and computer tools.
- It would be interesting to find a classical geometric proof!
- The next example shows an algebraic curve of the higher degree as a locus.

Searching for loci

Example

Let ABC be a triangle with a side AB and a vertex C on a circle k centered at A and radius $|AB|$. Find the locus of the orthocenter H of ABC when C moves along k.



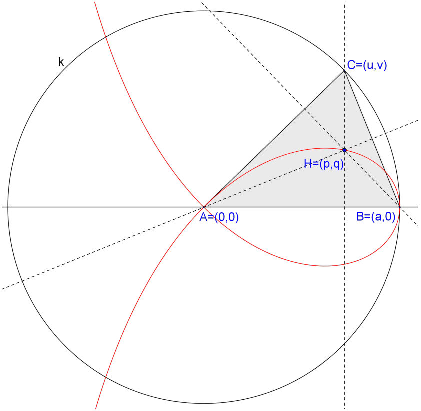
Searching for loci

Let $A = (0, 0)$, $B = (a, 0)$, $C = (u, v)$ and $H = (p, q)$. Then:

$(H - C) \perp (B - A) \Leftrightarrow h_1 : (p - u, q - v) \cdot (1, 0) = 0,$

$(H - A) \perp (C - B) \Leftrightarrow h_2 : (p, q) \cdot (u - a, v) = 0.$

$C \in k \Leftrightarrow h_3 : u^2 + v^2 - a^2 = 0.$



Searching for loci

Elimination of u, v in the system $h_1 = 0,\ h_2 = 0,\ h_3 = 0$ gives in the program Epsilon[2]

```
with(epsilon);
U:=[p-u,(u-a)p+vq,uu+vv-aa]:
X:=[p,q,u,v]:
CharSet(U,X);
```

the equation

$$p^2(a-p) - q^2(a+p) = 0$$

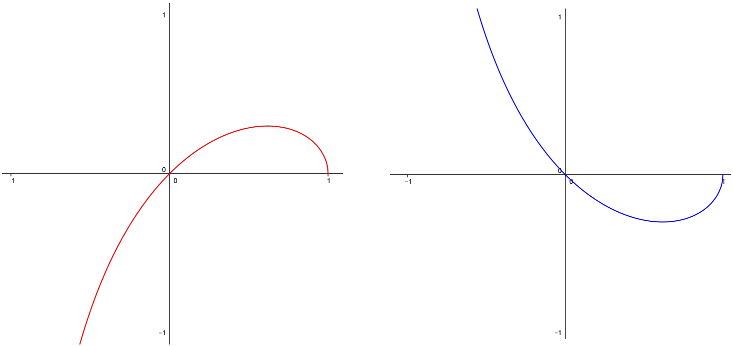
which is an algebraic curve of third degree called strophoid.

[2]Program Epsilon is freely distributed at `http://www-calfor.lip6.fr/~wang/epsilon/`

Searching for loci

The strophoid can be drawn as the union of graphs of two functions

$$f : q = p\sqrt{\frac{a-p}{a+p}} \qquad g : q = -p\sqrt{\frac{a-p}{a+p}}$$



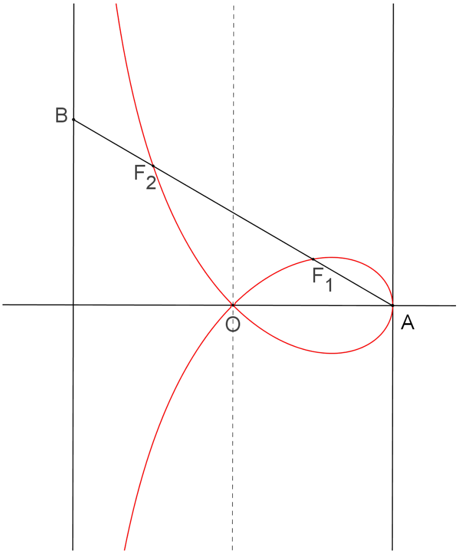
The graphs of the functions f and g for $a = 1$.

Searching for loci

- The strophoid or more exactly the right strophoid has many interesting properties.
- For instance you can use it in calculus to draw graphs above.
- Or to compute the area P of a loop which equals

$$P = 2a^2 - \frac{\pi a^2}{2}$$

  i.e. the area equals "two squares minus two circles over a".
- Strophoid is a candidate on the list of algebraic curves of degree higher than 3 which should be taught at schools.



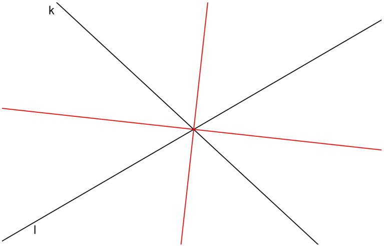
Strophoid is the locus of foci of the ellipse in a cylinder section when the section plane rotates around the tangent to the cylinder at A.

Searching for loci

3D locus example

Two skew lines k, l are given. Determine the locus of points which have the same distance to the lines k and l.

If two lines k, l intersect then it is well-known. The locus form two mutually orthogonal lines (in a plane) or two mutually orthogonal planes (in a space) which bisect angles of the lines.

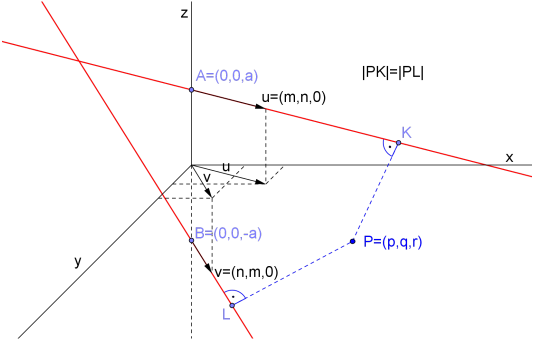


Searching for loci

What is the locus if the lines k, l are skew?

We determine the locus equation.

Let $k : X = A + t\vec{u}$ and $l : X = B + s\vec{v}$, where $A = [0, 0, a]$, $\vec{u} = (m, n, 0)$, $B = [0, 0, -a]$ and $\vec{v} = (n, m, 0)$. Choose m, n such that $m^2 + n^2 = 1$.



Searching for loci

Then

$$K \in k \Leftrightarrow h_1 := K - (A + t\vec{u}) = 0,$$

$$PK \perp k \Leftrightarrow h_2 := (P - K) \cdot \vec{u} = 0,$$

$$L \in l \Rightarrow h_3 := L - (B + s\vec{v}) = 0,$$

$$PL \perp l \Leftrightarrow h_4 := (P - L) \cdot \vec{v} = 0,$$

$$h_5 := |PK| - |PL| = 0,$$

$$h_6 := m^2 + n^2 - 1 = 0.$$

Searching for loci

Elimination of s, t, n in the system $h_1 = 0, h_2 = 0, \ldots, h_6 = 0$

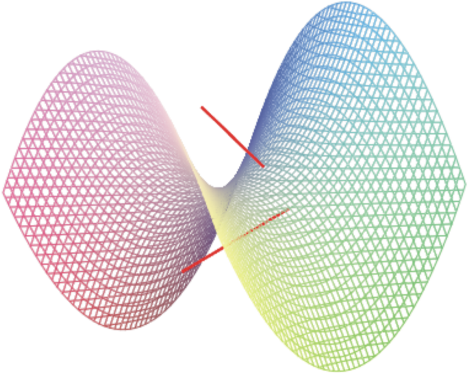
gives

$$p^2 - q^2 = cr,$$

where $c = -\frac{4a}{2m^2-1}$.

We see that this is the equation of an equilateral hyperbolic paraboloid.

Searching for loci

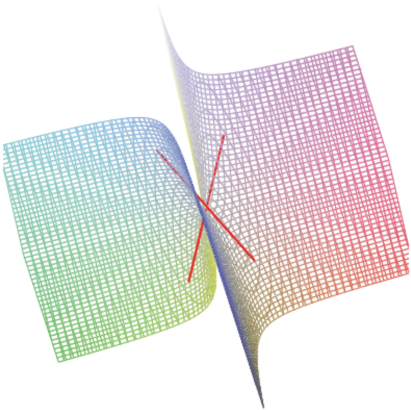


For $m = 1/2$ and $a = 1$ we get a hyperbolic paraboloid

$$p^2 - q^2 = 8r.$$

Searching for loci

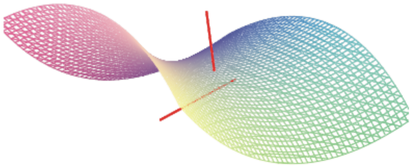
Note that the less is the coefficient a (i.e. the skew lines are at the smaller distance) the more is the hyperbolic paraboloid similar to two orthogonal planes. For instance for $a = 1/16$ and $m = 1/2$ we get



$$x^2 - y^2 = \frac{1}{2} r.$$

Searching for loci

If m and n tend to have equal direction (i.e. the skew lines tend to be parallel) then the hyperbolic paraboloid is similar to a plane.
For $a = 1$ and $m = 3/5$ we get



$$x^2 - y^2 = \frac{100}{7} r.$$

Verification in 3D

Verification in 3D

- Locus in a plane can be (numerically) verified either by ruler and compasses of by DGS.
- If the locus is in 3D then verification is more complicated.
- We'll verify the last locus by the method of descriptive geometry with the use of DGS.

Verification in 3D

- The locus of P which has the same distance to two given skew lines k, l is an equilateral hyperbolic paraboloid

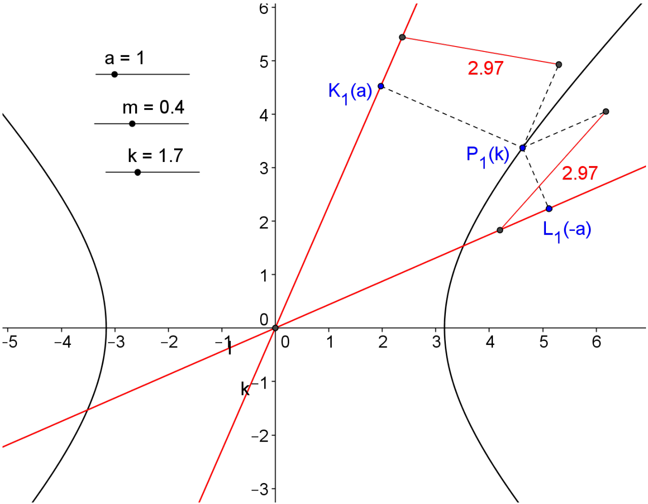
$$x^2 - y^2 = -\frac{4a}{2m^2 - 1}z.$$

- For a given a, m put $z = k$ and explore plane sections

$$x^2 - y^2 = -\frac{4a}{2m^2 - 1}k$$

  which are equilateral hyperbolas.
- In a one–plane orthogonal projection we map this equilateral hyperbola, place a point P on it, end construct feet K, L of perpendiculars to the skew lines k, l.
- Then we construct the distances $|PK|$ and $|PL|$.
- We show that $|PK| = |PL|$.

Verification in 3D



Verification in 3D

- Note the role of the equilateral hyperbola again.
- The method is based on the fact that a hyperbolic paraboloid can be covered by a pencil of conics.
- With sliders we can change the values m, a and k.
- This enables to verify the locus in all positions of P.

Thank you for attention